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Statistical Properties of Two-Dimensional Periodic Lorentz Gas with Infinite Horizon

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We study the asymptotic statistical behavior of the 2-dimensional periodic Lorentz gas with an infinite horizon. We consider a particle moving freely in the plane with elastic reflections from a periodic set of fixed convex scatterers. We assume that the initial position of the particle in the phase space is random with uniform distribution with respect to the Liouville measure of the periodic problem. We are interested in the asymptotic statistical behavior of the particle displacement in the plane as the time t goes to infinity. We assume that the particle horizon is infinite, which means that the length of free motion of the particle is unbounded. Then we show that under some natural assumptions on the free motion vector autocorrelation function, the limit distribution of the particle displacement in the plane is Gaussian, but the normalization factor is $(t \log t)^{1/2}$ and not $t^{1/2}$ as in the classical case. We find the covariance matrix of the limit distribution.

KEY WORDS: Lorentz gas; periodic configuration of scatterers; infinite horizon; corridors; statistical behavior of trajectories; "super"-diffusion; logarithmic corrections to square-root normalization; Gaussian limit distribution.

1. INTRODUCTION

In the present paper we investigate statistical properties of the two-dimensional Lorentz gas with a periodic configuration of scatterers. The Lorentz gas is an ensemble of noninteracting point particles which move freely with elastic reflections from fixed scatterers. It was introduced in the beginning of this century in works of Lorentz,[(1)] where it was assumed that the scatterers are randomly distributed in space. The Lorentz gas is a basic model

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for linearized kinetic equations.[2,3] Ergodic properties of the Lorentz gas were studied in works refs. 4–6.

In the present work we will consider the model of a Lorentz gas in which the scatterers are convex and they form a periodic (crystal) lattice in the plane. As a basic example, one may imagine a set of circular scatterers of a radius $a < 1/2$ centered at the sites of a square lattice with unit space (see Fig. 1). Since the particles in the Lorentz gas do not interact, they move independently. So let us consider the motion of one of the particles and assume that the initial position of the particle is distributed according to the Liouville measure of the periodic problem on the surface $|\mathbf{v}| = 1$. The main question we are interested in is the asymptotic behavior of typical (with respect to the Liouville measure) trajectories $\mathbf{x}(t)$ as $t \to \infty$. More precisely, the problem can be formulated as follows: What is the right normalization $N(t)$ such that there exists a limit distribution of $[\mathbf{x}(t) - \mathbf{x}(0)]/N(t)$ as $t \to \infty$ and what is the limit distribution?

This problem and the related one on the asymptotics of the velocity autocorrelation function have been studied intensively over the 10 years both theoretically and numerically (see refs. 8–19 and references cited there). It turns out that the situation is different for periodic configurations of scatterers with a "finite horizon" and for those with an "infinite horizon" ("without horizon" in another terminology).

By definition a periodic configuration of scatterers has a *finite horizon* if the length of free motion of the particle is bounded. Otherwise it has an *infinite horizon.* It is noteworthy that actually if a periodic configuration of scatterers has an infinite horizon, then there exist trajectories in which the particle does not reflect from the scatterers at all (see a more detailed description of such trajectories in the next section). In the example con-

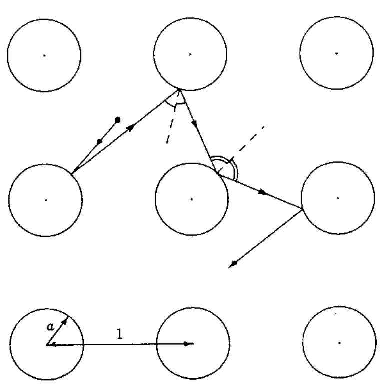

Fig. 1. Square lattice of circular scatterers.

sidered above of the square lattice of circular scatterers of radius $a<1/2$ the horizon is always infinite, since the lines $\{x=1/2\}$ and $\{y=1/2\}$ do not intersect any scatterer. To construct an example of a system with a finite horizon one may consider a configuration of circular scatterers centered at the sites of a triangular lattice with unit space. Then if the radius a of scatterers lies in the interval $\sqrt{3}/4<a<1/2$, the system has a finite horizon.

It was shown in refs. 7 and 8 that for any periodic configuration of scatterers with a finite horizon the velocity autocorrelation function satisfies the estimate

$$|\langle(\mathbf{v}(0), \mathbf{v}(t))\rangle| < C \exp(-\kappa t^{\gamma}) \tag{1.1}$$

with some C, $\kappa>0$ and $0<\gamma\leqslant 1$, where the average $\langle\cdot\rangle$ is taken with respect to the Liouville measure of the periodic problem and $(\mathbf{x}, \mathbf{y})$ means the scalar product of vectors $\mathbf{x}$ and $\mathbf{y}$. Numerical simulations[11] give the asymptotics

$$\langle(\mathbf{v}(0), \mathbf{v}(t_n))\rangle \sim (-1)^n \exp(-\kappa n^{\gamma}) \tag{1.2}$$

with $\gamma\approx 0.42$, $\kappa\approx 1.4$, where t_n is the moment of the nth reflection of the particle from the scatterers.

Denote

$$D(t)=\frac{\langle|\mathbf{x}(t)-\mathbf{x}(0)|^2\rangle}{4t} \tag{1.3}$$

Since

$$\mathbf{x}(t)-\mathbf{x}(0)=\int_0^t \mathbf{v}(s)\, ds$$

then

$$\begin{aligned} D(t) &= \frac{1}{4t}\int_0^t\int_0^t \langle(\mathbf{v}(s), \mathbf{v}(s'))\rangle\, ds\, ds' \\ &= \frac{1}{4t}\int_0^t\int_0^t \langle(\mathbf{v}(0), \mathbf{v}(s'-s))\rangle\, ds\, ds' \\ &= \frac{1}{2t}\int_0^t\int_0^s \langle(\mathbf{v}(0), \mathbf{v}(s'))\rangle\, ds'\, ds \end{aligned}$$

Integrating by parts, we get that

$$D(t)=\frac{1}{2}\int_0^t \langle(\mathbf{v}(0), \mathbf{v}(s))\rangle\, ds-\frac{1}{2t}\int_0^t s\langle(\mathbf{v}(0), \mathbf{v}(s))\rangle\, ds \tag{1.4}$$

so

$$D \equiv \lim_{t \to \infty} D(t) = \frac{1}{2} \int_0^\infty \langle (\mathbf{v}(0), \mathbf{v}(s)) \rangle \, ds \tag{1.5}$$

This is the celebrated Einstein–Green–Kubo formula for the diffusion coefficient (see, e.g., ref. 3). It shows that

$$D \equiv \lim_{t \to \infty} \frac{\langle |\mathbf{x}(t) - \mathbf{x}(0)|^2 \rangle}{4t} = \frac{1}{2} \int_0^\infty \langle (\mathbf{v}(0), \mathbf{v}(s)) \rangle \, ds \tag{1.6}$$

In ref. 8 (see also ref. 27) it was proved that for a periodic configuration of scatterers with a finite horizon, $\mathbf{x}(t) - \mathbf{x}(0)$ obeys the central limit theorem, i.e., the limit in distribution,

$$\lim_{t \to \infty} \frac{\mathbf{x}(t) - \mathbf{x}(0)}{\sqrt{t}} = \boldsymbol{\eta} \tag{1.7}$$

exists and $\boldsymbol{\eta}$ is a Gaussian random variable, $\langle \boldsymbol{\eta} \rangle = 0$. Moreover,

$$\mathbf{y}(s) = \lim_{t \to \infty} \frac{\mathbf{x}(st) - \mathbf{x}(0)}{\sqrt{t}}$$

is a Brownian process. It shows that for the periodic configuration of scatterers with a finite horizon, $\mathbf{x}(t) - \mathbf{x}(0)$ behaves at large t like a realization of a Brownian process.

In ref. 15 some heuristic arguments were given (see also ref. 9 for more refined arguments) which predict the asymptotics

$$\langle (\mathbf{v}(0), \mathbf{v}(t)) \rangle \sim \frac{\text{const}}{t} \tag{1.8}$$

for any periodic configuration of scatterers with an infinite horizon. Substituting this asymptotics into (1.4), one gets that

$$D(t) = \frac{\langle |\mathbf{x}(t) - \mathbf{x}(0)|^2 \rangle}{4t} \sim \text{const} \cdot \ln t \tag{1.9}$$

so that the diffusion coefficient $D = \lim_{t \to \infty} D(t)$ is infinite. One may expect from (1.9) that a right normalization for $[\mathbf{x}(t) - \mathbf{x}(0)]/N(t)$ for which it has a limit distribution is $N(t) = (t \ln t)^{1/2}$.[15,16,28]

In the present paper we show that this is really the case and our main result can be formulated as follows:

For any periodic configuration of scatterers with an infinite horizon the limit in distribution

$$\lim_{t \to \infty} \frac{\mathbf{x}(t) - \mathbf{x}(0)}{(t \ln t)^{1/2}} = \boldsymbol{\eta} \tag{1.10}$$

exists and $\boldsymbol{\eta}$ *is a Gaussian random variable.*

Moreover, we find a formula which expresses in geometric terms the covariance matrix of $\boldsymbol{\eta}$. The fundamental feature of the periodic configuration of scatterers with an infinite horizon is the existence of infinite corridors ("windows" in the terminology of ref. 16) along which a particle can move unboundedly long. It turns out that the faster running away of the particle to infinity than in the classical case is explained by the fact that the particle moves sometimes very long along the corridors. In numerical simulations one can see long "jumps" of the particle on the background of random Brownian-like motion.(19)

To understand better the nature of those jumps, let us consider the discrete dynamics of the particle. Let $\mathbf{x}_n = \mathbf{x}(t_n)$ be the position of the particle in the plane at the moment of the nth reflection. We are interested in the asymptotics of $\mathbf{x}_n - \mathbf{x}_0$ as $n \to \infty$ for typical trajectories. We can write that

$$\mathbf{x}_n - \mathbf{x}_0 = \sum_{j=0}^{n-1} (\mathbf{x}_{j+1} - \mathbf{x}_j) = \sum_{j=0}^{n-1} \mathbf{r}_j \tag{1.11}$$

which is the decomposition of the trajectory into segments of free motion. Hence

$$\begin{aligned} D_n &\equiv \frac{1}{4n} \langle |\mathbf{x}_n - \mathbf{x}_0|^2 \rangle \\ &= \frac{1}{4n} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \langle (\mathbf{r}_i, \mathbf{r}_j) \rangle \\ &= \frac{1}{4} \langle |\mathbf{r}_0|^2 \rangle + \frac{1}{2} \sum_{j=1}^{n-1} \langle (\mathbf{r}_0, \mathbf{r}_j) \rangle - \frac{1}{2n} \sum_{j=1}^{n-1} j \langle (\mathbf{r}_0, \mathbf{r}_j) \rangle \end{aligned} \tag{1.12}$$

so

$$D^0 = \lim_{n \to \infty} D_n = \frac{1}{4} \langle |\mathbf{r}_0|^2 \rangle + \frac{1}{2} \sum_{j=1}^{\infty} \langle (\mathbf{r}_0, \mathbf{r}_j) \rangle \tag{1.13}$$

This is a discrete analog of the Einstein–Green–Kubo formula (1.5).

It turns out (see refs. 16, 28, and 29 and Section 4 below) that because of the corridors the distribution function $P(R) = \Pr\{|\mathbf{r}_j| \leqslant R\}$ of the length of free motion has a powerlike tail at infinity,

$$1 - P(R) \sim \frac{\text{const}}{R^2} \tag{1.14}$$

(because of the invariance of the Liouville measure with respect to the dynamics, $\Pr\{|\mathbf{r}_j| > R\}$ does not depend on j). It implies that the variance of $|\mathbf{r}_j|$,

$$\langle |\mathbf{r}_j|^2 \rangle = \int_0^\infty R^2\, dP(R)$$

diverges logarithmically.

On the other hand, we prove in Section 5 that the correlation function $\langle (\mathbf{r}_i, \mathbf{r}_j) \rangle$ is finite for any $i \neq j$. Moreover, because of the strong ergodic properties of the system under consideration, one may expect that the correlation function is rapidly decreasing to zero when $|i-j| \to \infty$. So we deal in (1.11) with a sum of, in some sense, weakly dependent identically distributed random vectors $\mathbf{r}_j$ whose distribution has a powerlike tail (1.14). We discuss this important point in more detail in Section 6 and we show that the limit behavior of this sum as $n \to \infty$ is the same as if $\mathbf{r}_j$ were independent. Unfortunately, we do not have a full proof of this statement and our considerations in Section 6 are based on some natural conjectures concerning the character of the dependence of the vectors $\mathbf{r}_j$. Next we show that the logarithmic divergence of the variance of $\mathbf{r}_j$ leads to logarithmic corrections in the normalization of $\mathbf{x}_n - \mathbf{x}_0$ and we find that

$$\lim_{n \to \infty} \frac{\mathbf{x}_n - \mathbf{x}_0}{(n \ln n)^{1/2}} = \xi$$

where ξ is a Gaussian random vector variable. We get a formula for the covariance matrix of ξ which involves only some geometric characteristics of the lattice of scatterers.

The setup of the paper is the following. In Section 2 we introduce some geometrical notions which characterize the periodic configuration of scatterers with an infinite horizon. In Section 3 the discrete dynamics of billiard systems is discussed. In Section 4 we calculate the tail of the distribution of the free motion vectors and in Section 5 we estimate their correlation. In Section 6 we consider the limit distribution of the normalized trajectories in the discrete dynamics and in Section 7 we study it for the continuous-time dynamics. In Section 8 our main results are briefly summarized and more detailed calculations for square and triangular lattices of circular scatterers are presented.

2. CORRIDORS IN SYSTEMS WITH AN INFINITE HORIZON

In this section we give and recall some definitions and present some preliminary results concerning the Lorentz gas and billiards (see also, e.g., refs. 5, 20, and 24). A *scatterer* is an arbitrary, closed, bounded convex set Ω in $\mathbf{R}^2$ whose boundary $\partial\Omega$ is a C^4-smooth curve with bounded curvature radius. Let $\Omega_1,..., \Omega_N$ be an arbitrary finite set of scatterers in the plane and $\mathbf{e}_1$, $\mathbf{e}_2 \in \mathbb{R}^2$ be a basis in $\mathbb{R}^2$. We will call $\Omega_1,..., \Omega_N$ the *basic scatterers.* Denote $\mathbf{T}_i: \mathbf{x} \to \mathbf{x} + \mathbf{e}_i$ the shift in the vector $\mathbf{e}_i$, $i = 1, 2$, and let, for $\alpha = (j, m, n)$, where $1 \leqslant j \leqslant N$ and $m, n \in \mathbb{Z}$,

$$\Omega_\alpha = T_1^m T_2^n \Omega_j$$

be the shift of Ω_j in the vector $m\mathbf{e}_1 + n\mathbf{e}_2$. Then

$$\{\Omega_\alpha, \alpha \in A \equiv \{1,..., N\} \times \mathbb{Z}^2\}$$

is a *periodic configuration of scatterers with periods* $\mathbf{e}_1, \mathbf{e}_2$. In what follows we shall assume that the scatterers do not overlap, i.e., $\Omega_\alpha \cap \Omega_\beta = \varnothing$ if $\alpha \neq \beta$.

The parallelogram

$$D = \{\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \mid 0 \leqslant x_1, x_2 \leqslant 1\}$$

is called the *fundamental domain.* Without loss of generality we may assume that each basic scatterer Ω_j intersects with the fundamental domain.

The basic example of a periodic configuration of scatterers is a set of circles of some radius $a > 0$ with centers in the integer points $(m, n) \equiv m\mathbf{e}_1 + n\mathbf{e}_2$. In this example $N = 1$, i.e., there is one basic scatterer. Particular cases, which are applied usually in numerical simulations, are the *quadratic lattice of scatterers,* when $\mathbf{e}_1$, $\mathbf{e}_2$ form an orthonormal basis (see Fig. 1), and the *triangular lattice of scatterers,* when $|\mathbf{e}_1| = |\mathbf{e}_2|$ and $\mathbf{e}_1, \mathbf{e}_2$ form the angle $\pi/3$.

A periodic configuration of scatterers $\Omega = \{\Omega_\alpha, \alpha \in A\}$ is said to have a *finite horizon* if $L > 0$ exists such that any segment of the length L in the plane intersects some scatterer. This means that the length of free motion of the particle is bounded. Otherwise Ω is called a configuration with an *infinite horizon.*[30]

Let $\Omega = \{\Omega_\alpha, \alpha \in A\}$ be a periodic configuration of scatterers with an infinite horizon and $l = \{\omega_1 t\mathbf{e}_1 + \omega_2 t\mathbf{e}_2 + \mathbf{b},\ t \in \mathbb{R}\}$, $\omega_1, \omega_2 \in \mathbb{R}$, $\mathbf{b} \in \mathbb{R}^2$, be a straight line in $\mathbb{R}^2$ which intersects no scatterers. Denote $L = \{l\}$ the set of such lines.

Proposition 2.1. (i) The set $L = \{l\}$ is nonempty. (ii) Any $l \in L$ is rational in the sense that ω_1/ω_2 is rational or infinite. (iii) The set L is decomposed into a finite number of classes $L_{p/q}$ with the same value of $\omega_1/\omega_2 = p/q \in \mathbb{Q} \cup \{\infty\}$. (iv) $L_{p/q}$ consists of a countable number of open strips (corridors) such that any bounded domain in $\mathbb{R}^2$ intersects with only a finite number of such strips.

The proof of Proposition 2.1 is rather simple and we omit it. Now we will present without proof some results which describe the properties of corridors introduced in Proposition 2.1. We shall call two corridors C, C' equivalent iff $C' = T_1^m T_2^n C$ with some integer m, n. Let us enumerate all the corridors which touch the basic scatterer Ω_j by C_{jk}, $k = 1,..., N_j$; $j = 1,..., N$. We shall call $\{C_{jk}\}$ the *basic* corridors. (Remark that in principle a basic corridor can touch several basic scatterers, so it can be enumerated several times.) One can see easily that any corridor C is equivalent to some basic corridor.

It is clear that in corridors the particle can move unboundedly long. It turns out that also, on the contrary, each sufficiently long free path lies almost entirely in some corridor. Namely the following statement holds.

Proposition 2.2. There exist R_0, $R_1 > 0$ such that any finite segment of the length $R > R_0$ which intersects no scatterers lies entirely, except maybe for its extreme parts of total length less than R_1, in some corridor.

A simple corollary of Proposition 2.2 is the following statement.

Proposition 2.3. Consider in the plane a segment $[\mathbf{x}, \mathbf{y}]$ of free motion, $\mathbf{x} \in \partial\Omega_\alpha$, $\mathbf{y} \in \partial\Omega_\beta$. There exists $R_0 > 0$ such that if $|\mathbf{x} - \mathbf{y}| > R_0$, then Ω_α, Ω_β touch some corridor C from different sides of C. Moreover, for any $\varepsilon > 0$ one can choose R_0 in such a way that if $|\mathbf{x} - \mathbf{y}| > R_0$, then

$$|\mathbf{x} - \mathbf{x}_\alpha| < \varepsilon, \qquad |\mathbf{y} - \mathbf{x}_\beta| < \varepsilon$$

where $\mathbf{x}_\alpha$, $\mathbf{x}_\beta$ are the touching points of C and Ω_α, Ω_β, respectively.

Proposition 2.2 enables to prove also the following result.

Proposition 2.4. Each trajectory of the particle in the billiard either has no reflections from the scatterers (then it lies entirely in some corridor) or has infinitely many reflections from the scatterers in both directions of time.

3. DISCRETE DYNAMICS

Let us recall some definitions and results from the billiard theory.[5,20–23] We introduce first a natural coordinate at the boundary of each scatterer. To this end let us fix an arbitrary point $O_j \in \partial\Omega_j$ at the boundary of each basic scatterer and fix the point $O_\alpha = T_1^m T_2^n O_j$ at $\partial\Omega_\alpha$ for $\alpha = (j, m, n)$. For $x \in \partial\Omega_\alpha$ a *natural coordinate* of x is the length of the arc $O_\alpha x$ in the counterclockwise direction.

Consider a trajectory $\mathbf{x}(t)$ of the particle in $\mathbb{R}^2 \setminus \bigcup_{\alpha \in A} \Omega_\alpha$, which has reflections from the scatterers (see Proposition 2.4). Denote $\{t_n, -\infty < n < \infty\}$ the time moments of the reflections, in such a way that

$$\cdots < t_{-1} < t_0 \leqslant 0 < t_1 < t_2 < \cdots$$

The nth reflection is characterized by three quantities: $\alpha_n = \alpha(t_n) \in A$, the index of the scatterer; $s_n = s(t_n)$, the natural coordinate of the reflection point; and $\eta_n = \eta(t_n)$, the reflection angle (the angle with the sign between the velocity vector $\mathbf{v}$ after the reflection and the outer normal vector $\mathbf{n}$ at the reflection point). It is clear that the triple $\lambda_n = (\alpha_n, s_n, \eta_n)$ determines the whole trajectory uniquely. In particular, it determines the triple $\lambda_{n+1} = (\alpha_{n+1}, s_{n+1}, \eta_{n+1})$ of the next reflection. The map

$$T: \quad (\alpha_n, s_n, \eta_n) \to (\alpha_{n+1}, s_{n+1}, \eta_{n+1}) \tag{3.1}$$

is the Poincaré section map. It acts in the phase space of triples

$$\Lambda = \bigcup_{\alpha \in A} \left(\{0 \leqslant s < |\partial\Omega_\alpha|\} \times \left\{ -\frac{\pi}{2} \leqslant \eta \leqslant \frac{\pi}{2} \right\} \right) \tag{3.2}$$

$T: \Lambda \to \Lambda$ and it determines the discrete dynamics: $\lambda_n = T^n \lambda_0$.

Define the involution

$$S: \quad (\alpha, s, \eta) \to (\alpha, s, -\eta) \tag{3.3}$$

It is easy to see that S is a *symmetry* for T in the sense that

$$S^2 = STST = \mathrm{Id} \tag{3.4}$$

(the identity map), so T is invertible,

$$T^{-1} = STS$$

Another important property of T is that it *preserves the Liouville measure* $\cos\eta \, ds \, d\eta$, i.e.,

$$\sum_{\alpha \in A} \iint_{T^{-1}(B)} \cos\eta \, ds \, d\eta = \sum_{\alpha \in A} \iint_B \cos\eta \, ds \, d\eta \tag{3.5}$$

for any finite measurable set $B \subset \Lambda$.

Consider the map $i: \mathbb{R}^2 \to \mathbb{T}^2$, which identifies all the points $\{\mathbf{x} + m\mathbf{e}_1 + n\mathbf{e}_2,\ m,\ n \in \mathbb{Z}\}$ with $\mathbf{x}$. In particular, i identifies all the scatterers $\{\Omega_{(j,m,n)},\ m,\ n \in \mathbb{Z}\}$ with Ω_j, $j = 1, \ldots, N$, so we can consider the scatterers $\Omega_1, \ldots, \Omega_N$ in $\mathbb{T}^2$ and a billiard system in $\mathbb{T}^2 \setminus \bigcup_{1 \leqslant j \leqslant N} \Omega_j$. It is called the *periodic billiard system*, or the *Sinai Billiard.*[20] If $\mathbf{x}(t) = x_1(t)\,\mathbf{e}_1 + x_2(t)\,\mathbf{e}_2$ is a trajectory of the particle in the original billiard $\mathbb{R}^2 \setminus \bigcup_{\alpha \in A} \Omega_\alpha$, then $i(\mathbf{x}(t)) = \{x_1(t)\}\,\mathbf{e}_1 + \{x_2(t)\}\,\mathbf{e}_2$ is the corresponding trajectory in the periodic billiard.

The phase space of the periodic billiard is

$$A_0 = \bigcup_{j=1}^{N} \left(\{0 \leqslant s < |\partial\Omega_j|\} \times \left\{ -\frac{\pi}{2} \leqslant \eta \leqslant \frac{\pi}{2} \right\} \right) \tag{3.6}$$

which is the union of N cylinders, and the Poincaré map $T_0: A_0 \to A_0$ is determined by the condition that if $T: (\alpha, s, \eta) \to (\alpha', s', \eta')$, where $\alpha = (j, m, n)$, $\alpha' = (j', m', n')$, then $T_0: (j, s, \eta) \to (j', s', \eta')$. The Poincaré map preserves the probability Liouville measure

$$\mu_0(ds\, d\eta) = Z^{-1} \cos n\, ds\, d\eta$$

$$Z = \sum_{j=1}^{N} \int_{\partial\Omega_j} \int \cos \eta\, d\eta\, ds = 2 \sum_{j=1}^{N} |\partial\Omega_j| \tag{3.7}$$

on A_0. In what follows we shall use the notation

$$\langle f(\lambda) \rangle \equiv \int_{A_0} f(\lambda)\, \mu_0(ds\, d\eta) = Z^{-1} \sum_{j=1}^{N} \iint f(\lambda) \cos \eta\, ds\, d\eta, \quad \lambda = (j, s, \eta)$$

A basic result of the billiard theory[5,20–23] is that T_0 possesses strong ergodic properties with respect to μ_0. This implies, in particular, that for typical trajectories after n reflections, as $n \to \infty$, the fraction of the reflections belonging to a given domain $B \subset A_0$ of the phase space tends to the area of B with respect to the measure μ_0.

For the discrete dynamics the problem we are interested in can be formulated in the following way. Denote $\mathbf{x}(\lambda) = \mathbf{x}(\alpha, s, \eta) \in \mathbb{R}^2$ the point in $\partial\Omega_\alpha$ with the natural coordinate s (obviously it does not depend on η). What is the asymptotic behavior of $\mathbf{x}(T^n\lambda) - \mathbf{x}(\lambda)$, when $n \to \infty$, with respect to the Liouville measure? More precisely, let us remark that $\mathbf{x}(T^n\lambda) - \mathbf{x}(\lambda)$ can be viewed as a vector function on A_0 in the sense that if $\lambda = (\alpha, s, \eta)$, where $\alpha = (j, m, n)$, then $\mathbf{x}(T^n\lambda) - \mathbf{x}(\lambda)$ does not depend on m, n. Then the problem is: What is the asymptotic behavior of $\langle |\mathbf{x}(T^n\lambda) - \mathbf{x}(\lambda)| \rangle$ when $n \to \infty$, and what is the limit distribution when $n \to \infty$ (if it does exist) of

$$\frac{\mathbf{x}(T^n\lambda) - \mathbf{x}(\lambda)}{\langle |\mathbf{x}(T^n\lambda) - \mathbf{x}(\lambda)| \rangle} \tag{3.8}$$

where λ is μ_0-distributed on the phase space Λ_0? In a more general framework one may think of an absolutely continuous distribution $\gamma(d\lambda)$ for λ. Then an additional question is whether the limit distribution of the vector (3.8) is independent of γ.

Denote

$$\mathbf{r}(\lambda) = \mathbf{x}(T\lambda) - \mathbf{x}(\lambda)$$

Remark that $\mathbf{r}(\lambda)$ is the vector of free motion between points $\mathbf{x}(\lambda)$ and $\mathbf{x}(T\lambda)$. As we noticed before, $\mathbf{r}(\lambda)$ can be viewed as a vector function on the phase space Λ_0. Now we can represent $\mathbf{x}(T^n\lambda) - \mathbf{x}(\lambda)$ as

$$\mathbf{x}(T^n\lambda) - \mathbf{x}(\lambda) = \sum_{j=0}^{n-1} \mathbf{r}(T^j\lambda) \tag{3.9}$$

which is obviously the decomposition of $\mathbf{x}(T^n\lambda) - \mathbf{x}(\lambda)$ into the sum of vectors of free motion. Since $\mathbf{r}$ is a function on Λ_0, we can change in the last formula T^j for T_0^j, so we get that

$$\mathbf{x}(T^n\lambda) - \mathbf{x}(\lambda) = \sum_{j=0}^{n-1} \mathbf{r}(T_0^j\lambda) \tag{3.10}$$

Thus, the problem of the statistical properties of the vector $\mathbf{x}(T^n\lambda) - \mathbf{x}(\lambda)$, when $n \to \infty$, is just the problem of the limit theorem for the vector function of free motion $\mathbf{r}(\lambda)$ with respect to the measure-preserving map T_0.

In the following two sections we shall study some properties of the distribution of $\mathbf{r}(\lambda)$ with respect to the Liouville measure $\mu_0(d\lambda)$ and of the second moment $\langle(\mathbf{r}(T^j\lambda), \mathbf{r}(\lambda))\rangle$. Remark that because of an infinite horizon, the vector of free motion $\mathbf{r}(\lambda)$ is unbounded and the important question for us is the asymptotics of the tail of the distribution of $\mathbf{r}(\lambda)$ with respect to the Liouville measure.

4. DISTRIBUTION OF THE FREE MOTION VECTOR

Denote the distribution of the vector of free motion $\mathbf{r}(\lambda)$ with respect to the Liouville measure μ_0 by ν^0.

Proposition 4.1.[20,30] The distribution ν^0 is symmetric.

Proof. Since $\mathbf{x}(S\lambda) = \mathbf{x}(\lambda)$, then, by (3.4),

$$\mathbf{r}(ST\lambda) = \mathbf{x}(TST\lambda) - \mathbf{x}(ST\lambda) = \mathbf{x}(S\lambda) - \mathbf{x}(ST\lambda) = \mathbf{x}(\lambda) - \mathbf{x}(T\lambda) = -\mathbf{r}(\lambda)$$

but the distribution of $\mathbf{r}(ST\lambda)$ coincides with that of $\mathbf{r}(\lambda)$, because both S and T preserve the measure μ_0, hence $\mathbf{r}(\lambda) = -\mathbf{r}(\lambda)$ in distribution, which proves Proposition 4.1.

Let us turn now to the computation of the tail of the distribution of $\mathbf{r}(\lambda)$. Consider a long segment $[\mathbf{x}, \mathbf{y}]$ of free motion, $\mathbf{x} = \mathbf{x}(\lambda) \in \partial\Omega_j$, $\mathbf{y} = \mathbf{x}(T\lambda) \in \partial\Omega_\beta$. By the Corollary to Proposition 2.2, there is some corridor C_{jk} such that Ω_j and Ω_β touch C_{jk} from different sides of C_{jk} and

$$|\mathbf{x} - \mathbf{x}_{jk}| \ll 1, \qquad |\mathbf{y} - \mathbf{x}_\beta| \ll 1$$

where $\mathbf{x}_{jk}, \mathbf{x}_\beta$ are the touching points of C_{jk} and Ω_j, Ω_β, respectively. In such a case we shall say that the segment $[\mathbf{x}, \mathbf{y}] = [\mathbf{x}(\lambda), \mathbf{x}(T\lambda)]$ *belongs* to the corridor C_{jk}; the notation is $[\mathbf{x}(\lambda), \mathbf{x}(T\lambda)] \in C_{jk}$.

Denote by d_{jk} the width of the corridor C_{jk} (i.e., the distance between its boundary lines) and by

$$\boldsymbol{\omega}_{jk} = \omega_{jk1}\mathbf{e}_1 + \omega_{jk2}\mathbf{e}_2$$

the unit vector which is parallel to the boundary line of the corridor C_{jk}. Let $\mathbf{x}_{jk} \in \partial\Omega_j \cap \partial C_{jk}$ be the touching point of Ω_j and C_{jk}, and $\mathbf{n}_{jk}$ be the vector of the outer normal to $\partial\Omega_{jk}$ at $\mathbf{x}_{jk}$. To choose $\boldsymbol{\omega}_{jk}$ in a unique way, we shall assume that $\boldsymbol{\omega}_{jk}$ is in the counterclockwise direction from $\mathbf{n}_{jk}$ (see Fig. 2). Let $\boldsymbol{\omega}_{jk}^{\pm} = \pm\boldsymbol{\omega}_{jk}$.

Since by Proposition 2.1 $\omega_{jk1}/\omega_{jk2}$ is rational or infinite, C_{jk} touches a periodic set of scatterers. Let $\Omega_{\alpha(j,k,\pm)}$ be the closest scatterer to Ω_j in the direction of $\omega_{jk}^{\pm}$ which touches the corridor C_{jk} from the same side as Ω_j does. Denote by $h_{jk}^{\pm}$ the distance between the touching points of the scatterers Ω_j and $\Omega_{\alpha(j,k,\pm)}$ with the corridor C_{jk}.

Let us compute the asymptotics of the probability

$$p_{jk}^{\pm}(R) = \Pr\{|\mathbf{r}(\lambda)| > R, \mathbf{x}(\lambda) \in \partial\Omega_j, [\mathbf{x}(\lambda), \mathbf{x}(T\lambda)] \in C_{jk}, (\mathbf{r}(\lambda), \boldsymbol{\omega}_{jk}^{\pm}) > 0\} \tag{4.1}$$

when $R \to \infty$.

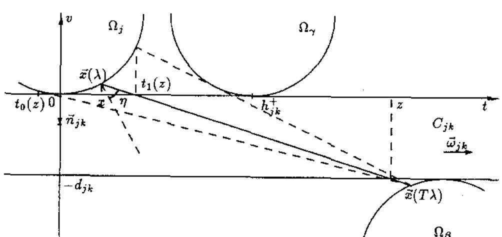

Fig. 2. A long segment of free motion.

Proposition 4.2. For $R \to \infty$,

$$p_{jk}^{\pm}(R) = \frac{1}{4 \sum_{j=1}^{N} |\partial \Omega_j|} h_{jk}^{\pm} \left(\frac{d_{jk}}{R}\right)^2 + O(R^{-5/2}) \tag{4.2}$$

Proof. Let us consider for the sake of definiteness the asymptotics of $p_{jk}^{+}(R)$. To simplify notations we write $d = d_{jk}$, $h = h_{jk}^{+}$. Let $\mathbf{x}(\lambda) \in \partial\Omega_j$ be a point in a small neighborhood of $\mathbf{x}_j$. Let us introduce an orthonormal basis $\mathbf{f}_1, \mathbf{f}_2$ in $\mathbb{R}^2$ such that $\mathbf{f}_1 = \boldsymbol{\omega}_{jk}$ and consider a coordinate system (t, v) with this basis and with the origin at $\mathbf{x}_{jk}$. We will assume for the sake of definiteness that Ω_j lies in the half-plane $\{v \geqslant 0\}$. Let $(z, -d_{jk})$ be the intersection point of the segment $[\mathbf{x}(\lambda), \mathbf{x}(T\lambda)]$ with the line $\{v = -d_{jk}\}$ (see Fig. 2). Then

$$z = |\mathbf{x}(T\lambda) - \mathbf{x}(\lambda)| + O(1) = |\mathbf{r}(\lambda)| + O(1)$$

and so, to prove (4.2), it is enough to prove a similar asymptotics for the distribution function of z. Let us compute the latter. We have

$$\frac{z - t}{d + f(t)} = \tan[\eta + \arctan f'(t)] \tag{4.3}$$

where $\mathbf{x}(\lambda) = (t, v)$; $v = f(t)$ is the equation of $\partial\Omega_j$ in the vicinity of $\mathbf{x}_{jk}$ and η is the reflection angle (see Fig. 2). After some calculations we get from (4.3) that

$$\cos\eta \, d\eta = \frac{[1 + wf'(t)] \, dz}{\{1 + [f'(t)]^2\}^{1/2} (1 + w^2)^{3/2} [d + f(t)]}$$

where

$$w = \frac{z - t}{d + f(t)}$$

Hence, the density of probability of the distribution of z with respect to the Liouville measure $Z^{-1} \cos\eta \, d\eta \, ds$ is equal to

$$p(z) = \int_{t_0(z)}^{t_1(z)} Z^{-1} \frac{1 + wf'(t)}{\{1 + [f'(t)]^2\}^{1/2} (1 + w^2)^{3/2} [d + f(t)]} \frac{ds}{dt} \, dt$$

where $t_0(z) = \min_\eta t$, $t_1(z) = \max_\eta t$ under fixed z (see Fig. 2). Since $ds/dt = \{1 + [f'(t)]^2\}^{1/2}$,

$$p(z) = Z^{-1} \int_{t_0(z)}^{t_1(z)} \frac{1 + wf'(t)}{(1 + w^2)^{3/2} [d + f(t)]} \, dt$$

Now,

$$w=\frac{z}{d}+O(1),\quad t_0(z)=O(z^{-1}),\quad t_1(z)=O(z^{-1/2}),\quad f(t)=O(t^2)$$

so

$$p(z)=Z^{-1}\frac{1}{[1+(z/d)^2]^{3/2}\,d}\int_{t_0(z)}^{t_1(z)}[1+wf'(t)]\,dt[1+O(z^{-1})] \tag{4.4}$$

Furthermore,

$$\int_{t_0(z)}^{t_1(z)}[1+wf'(t)]\,dt=\frac{z}{d}f(t_1(z))+O(z^{-1/2})$$

A simple geometrical calculation gives that

$$f(t_1(z))=\frac{hd}{z}[1+O(z^{-1})]$$

so that

$$\int_{t_0(z)}^{t_1(z)}[1+wf'(t)]\,dt=h+O(z^{-1/2})$$

Substituting this formula into (4.4), we get that

$$p(z)=Z^{-1}\frac{hd^2}{z^3}[1+O(z^{-1/2})]$$

Hence

$$\Pr\{z\geqslant R\}=\int_R^\infty p(z)\,dz=Z^{-1}h\frac{d^2}{2R^2}[1+O(R^{-1/2})]$$

Proposition 4.2 is proved.

It is noteworthy that for large R the distribution $-dp_{jk}^+(R)$ is localized near points of a periodic lattice (by periodic lattice we mean here a locally finite periodic set on the line). Indeed, according to the Corollary to Proposition 2.2, for large $|\mathbf{y}-\mathbf{x}|$, $\mathbf{x}$ is close to $\mathbf{x}_{jk}$ and $\mathbf{y}$ is close to $\mathbf{x}_\beta$, the touching point of the corridor C_{jk} and the scatterer Ω_β, which contains $\mathbf{y}$, so $|\mathbf{y}-\mathbf{x}|$ is close to $|\mathbf{x}_\beta-\mathbf{x}_{jk}|$. Since the scatterers Ω_β form a periodic set at the boundary of the corridor, then for large $|\mathbf{x}_\beta-\mathbf{x}_{jk}|$ this quantity is close to a periodic lattice, so $|\mathbf{y}-\mathbf{x}|$ is also close to that lattice. It means that for large R the distribution $-dp_{jk}^{+}(R)$ is localized near the periodic lattice. As a consequence, the corresponding distribution density has δ-shaped peaks near the points of that lattice, so it does not have the regular powerlike asymptotics

$$-\frac{dp_{jk}^{+}(R)}{dR}=\frac{\text{const}}{R^3}+o\left(\frac{1}{R^3}\right)$$

in spite of the fact that the distribution function $p_{jk}^{+}(R)$ obeys (4.2).

Denote

$$d=\max_{1\leqslant j\leqslant N}\ \max_{1\leqslant k\leqslant N_j}\ d_{jk}$$

and consider the strips

$$S_{jk}=\{\mathbf{x}\in\mathbb{R}^2\,|\,\text{dist}(\mathbf{x},l_{jk})<d\}$$

around the lines

$$l_{jk}=\{\mathbf{x}\in\mathbb{R}^2\,|\,\mathbf{x}=t\boldsymbol{\omega}_{jk},\ t\in\mathbb{R},$$

which go in the corridor directions $\boldsymbol{\omega}_{jk}$. Let

$$S=\bigcup_{j=1}^{N}\ \bigcup_{k=1}^{N_j}\ S_{jk}$$

Then by the Corollary of Proposition 4.2, $\mathbf{r}(\lambda)\in S$ if $R=|\mathbf{r}(\lambda)|$ is sufficiently large. This means that the support of the distribution $v^0(d\mathbf{r})$ of $\mathbf{r}(\lambda)$ resembles an "octopus," with $2M$ legs stretched out in the corridor directions $\boldsymbol{\omega}_{jk}^{\pm}$, where M is the total number of different corridor directions.

Proposition 4.3. $\langle|\mathbf{r}(\lambda)|\rangle<\infty$, $\langle\mathbf{r}(\lambda)\rangle=0$, $\langle|\mathbf{r}(\lambda)|^2\rangle=\infty$. Moreover, if $\phi_R(\mathbf{x})=|\mathbf{x}|^2$ for $|\mathbf{x}|<R$, $\phi_R(\mathbf{x})=0$ for $|\mathbf{x}|\geqslant R$, then

$$\langle\phi_R(\mathbf{r}(\lambda))\rangle=C_0\ln R+O(1) \tag{4.5}$$

when $R\to\infty$.

Remark. This means that the variance of $\mathbf{r}(\lambda)$ diverges logarithmically.

Proof. We have

$$\langle\mathbf{r}(\lambda)\rangle=\int\mathbf{r}v^0(d\mathbf{r})=0$$

because ν^0 is symmetric. Next, for large $R_0 > 0$,

$$\int_{|\mathbf{r}| > R_0} |\mathbf{r}|\, \nu^0(d\mathbf{r}) = \sum_{j=1}^{N} \sum_{k=1}^{N_j} \sum_{\pm} \int_{R_0}^{\infty} R[-dp_{jk}^{\pm}(R)]$$

$$= \sum_{j=1}^{N} \sum_{k=1}^{N_j} \sum_{\pm} \left[R_0\, p_{jk}^{\pm}(R_0) + \int_{R_0}^{\infty} p_{jk}^{\pm}(R)\, dR \right] < \infty$$

because of the asymptotics (4.2). Similarly, for $R_1 \to \infty$,

$$\int_{R_1 > |\mathbf{r}| > R_0} |\mathbf{r}|^2\, \nu^0(d\mathbf{r})$$

$$= \sum_{j=1}^{N} \sum_{k=1}^{N_j} \sum_{\pm} \int_{R_0}^{R_1} R^2[-dp_{jk}^{\pm}(R)]$$

$$= \sum_{j=1}^{N} \sum_{k=1}^{N_j} \sum_{\pm} \left\{ [R_0^2\, p_{jk}^{\pm}(R_0) - R_1^2\, p_{jk}^{\pm}(R_1)] + 2 \int_{R_0}^{R_1} R p_{jk}^{\pm}(R)\, dR \right\}$$

$$= \text{const} \ln R_1 + O(1)$$

which proves (4.5). Proposition 4.3 is proved.

In the next theorem we calculate the singularity at the origin of the characteristic function $\chi(\mathbf{t})$ of the free motion vector $\mathbf{r}(\lambda)$.

Theorem 4.4. For $\mathbf{t} \to 0$, $\mathbf{t} \in \mathbb{R}^2$:

$$\chi(\mathbf{t}) \equiv \langle \exp[i(\mathbf{t}, \mathbf{r}(\lambda))] \rangle$$

$$= \exp\left[\sum_{j=1}^{N} \sum_{k=1}^{N_j} \alpha_{jk} (\mathbf{t}, \boldsymbol{\omega}_{jk})^2 \ln |(\mathbf{t}, \boldsymbol{\omega}_{jk})| + (A\mathbf{t}, \mathbf{t}) + O(|\mathbf{t}|^{9/4}) \right]$$

where

$$\alpha_{jk} = \frac{1}{4 \sum_{l=1}^{N} |\partial \Omega_l|}\, h_{jk}\, d_{jk}^2, \qquad h_{jk} = h_{jk}^{+} + h_{jk}^{-}$$

and $(A\mathbf{t}, \mathbf{t})$ is a quadratic form.

Proof. We have

$$\chi(\mathbf{t}) = \langle \exp[i(\mathbf{t}, \mathbf{r}(\lambda))] \rangle$$

$$= \int \exp[i(\mathbf{t}, \mathbf{r})]\, \nu^0(d\mathbf{r})$$

$$= \int_{|\mathbf{r}| \leqslant R} \exp[i(\mathbf{t}, \mathbf{r})]\, \nu^0(d\mathbf{r}) + \int_{|\mathbf{r}| > R} \exp[i(\mathbf{t}, \mathbf{r})]\, \nu^0(d\mathbf{r})$$

$$\equiv \chi_0(\mathbf{t}; R) + \chi_1(\mathbf{t}; R)$$

The function $\chi_0(\mathbf{t}; R)$ is analytic for any fixed R. Compute the singularity of $\chi_1(\mathbf{t}; R)$ at the origin.

As we mentioned above, supp $\nu^0(d\mathbf{r})$ outside of a circle $\{|\mathbf{r}| > R\}$ for large R consists of several components, "octopus legs," corresponding to different basic corridors. It enables us to represent $\chi_1(\mathbf{t}; R)$ as

$$\chi_1(\mathbf{t}; R) = \sum_{j=1}^{N} \sum_{k=1}^{N_j} \sum_{\pm} \chi_{jk}^{\pm}(\mathbf{t}; R) \tag{4.6}$$

where

$$\chi_{jk}^{\pm}(\mathbf{t}; R) = \int_{S_{jk}^{\pm}(R)} \exp[i(\mathbf{t}, \mathbf{r})]\, \nu_{jk}^{\pm}(d\mathbf{r}; R) \tag{4.7}$$

where

$$S_{jk}^{\pm}(R) = \{\mathbf{r} \in S_{jk} \mid \pm(\mathbf{r}, \omega_{jk}) > 0, |\mathbf{r}| > R\}$$

and

$$\nu_{jk}^{\pm}(A; R) = \Pr\{\mathbf{r}(\lambda) \in A, \lambda \in V_{jk}^{\pm}(R)\}$$

where

$$V_{jk}^{\pm}(R) = \{(s, \eta) \mid \text{for } \lambda = (j, s, \eta): \\ |\mathbf{r}(\lambda)| > R, \mathbf{x}(\lambda) \in \partial\Omega_j, [\mathbf{x}(\lambda), \mathbf{x}(T\lambda)] \in C_{jk}, \pm(\mathbf{r}(\lambda), \omega_{jk}) > 0\}$$

Remark that (4.6) is a sum over the basic corridors and the $\pm$ directions of those corridors.

Let us compute the singularity of $\chi_{jk}^{+}(\mathbf{t}; R)$ when $\mathbf{t} \to 0$. Let $\mathbf{r} = (x, y)$, where x, y are coordinates in the orthonormal basis $\mathbf{f}_1 = \boldsymbol{\omega} jk$, $\mathbf{f}_2$ with the origin at $\mathbf{x}_{jk} = \partial\Omega_j \cap \partial C_{jk}$. Denote

$$P(x, y) = \int_x^\infty \int_y^\infty \nu_{jk}^{+}(dx\, dy)$$

For the sake of brevity we do not indicate the dependence of $P(x, y)$ on j, k. By (4.7)

$$\chi_{jk}^{+}(\mathbf{t}; R) = \int_{S_{jk}^{+}(R)} \exp[i(\mathbf{t}, \mathbf{r}]\, \nu_{jk}^{+}(d\mathbf{r})$$
$$= \int_{S_{jk}^{+}(R)} \exp[i(ux + vy)]\, \frac{d^2P(x, y)}{dx\, dy}\, dx\, dy$$

Consider the domain

$$U = \{(x, y) \in \mathbb{R}^2 \mid |y - d_{jk}| < |x|^{-0.99}\}$$

We state that

$$\operatorname{supp} v_{jk}^{+}(\cdot; R) \subset U \tag{4.8}$$

(if R is large enough). We use here again that any long segment of free motion lies almost entirely in a basic corridor. A simple geometrical calculation shows that the condition that $[\mathbf{x}(\lambda), \mathbf{x}(T\lambda)]$ does not intersect the closest scatterer Ω_{γ} implies that

$$|y(\lambda)| < \text{const} \cdot |\mathbf{r}(\lambda)|^{-1}$$

where

$$\mathbf{x}(\lambda) = (x(\lambda), y(\lambda)), \qquad \mathbf{r}(\lambda) = \mathbf{x}(T\lambda) - \mathbf{x}(\lambda)$$

Similarly,

$$|y(T\lambda) - d_{jk}| < \text{const} \cdot |\mathbf{r}(\lambda)|^{-1}$$

so

$$|y(T\lambda) - y(\lambda) - d_{jk}| < \text{const} \cdot |\mathbf{r}(\lambda)|^{-1}$$

which implies (4.8). Remark that (4.8) means that the support of $v_{jk}^{+}(\cdot; R)$ lies in a narrow region around the line $y = d_{jk}$.

By (4.8) for $x > R$,

$$P(x, y) = 0 \qquad \text{if} \quad y > d_{jk} + x^{-0.99}$$

$$P(x, y) = P_0(\mathrm{x}) \qquad \text{if} \quad y < d_{jk} - x^{-0.99}$$

where

$$P_0(x) = \Pr\{\mathbf{x}(\lambda) \in \partial\Omega_j, [\mathbf{x}(\lambda), \mathbf{x}(T\lambda)] \in C_{jk}, (\mathbf{r}(\lambda), \boldsymbol{\omega}_{jk}) > x\}$$

Remark that, by Proposition 4.2 for $x \to \infty$,

$$P_0(x) = \frac{\alpha_{jk}^{+}}{x^2} + O(x^{-5/2}) \tag{4.9}$$

Define in the half-plane $\{(x, y) \mid x \geqslant R\}$ an auxiliary function $Q(x, y)$ such that

$$Q(x, y) = \begin{cases} 0 & \text{if} \quad y \geqslant d_{jk} \\ P_0(x) & \text{if} \quad y < d_{jk} \end{cases} \tag{4.10}$$

and decompose $\chi_{jk}^{+}(\mathbf{t}; R)$ into the sum of three terms,

$$\chi_{jk}^{+}(\mathbf{t}; R) = \alpha(\mathbf{t}; R) + \beta(\mathbf{t}; R) + \gamma(\mathbf{t}; R) \tag{4.11}$$

where

$$\alpha(\mathbf{t}; R) = \iint_{\{x \geqslant R\}} \exp[i(ux+vy)] \frac{d^2Q(x, y)}{dx\, dy}\, dx\, dy$$

$$\beta(\mathbf{t}; R) = \iint_{\{x \geqslant R\}} \exp[i(ux+vy)] \frac{d^2(P(x, y)-Q(x, y))}{dx\, dy}\, dx\, dy$$

$$\gamma(\mathbf{t}; R) = \iint_{S_{jk}^{+}(R)\setminus\{x \geqslant R\}} \exp[i(ux+vy)] \frac{d^2P(x, y)}{dx\, dy}\, dx\, dy$$

Notice that since $S_{jk}^{+}(R)\setminus\{x \geqslant R\}$ is compact, then $\gamma(\mathbf{t}; R)$ is analytic in $\mathbf{t}$. Let us compute singularities of $\alpha(\mathbf{t}; R)$ and $\beta(\mathbf{t}; R)$ at $\mathbf{t}=0$. We have by (4.10) that

$$\alpha(\mathbf{t}; R) = -\int_R^\infty \exp[i(ux+vd_{jk})] \frac{dP_0(x)}{dx}\, dx$$

$$= -\exp[i(uR+vd_{jk})]\, P_0(x) + \int_R^\infty (iu) \exp[i(ux+vd_{jk})]\, P_0(x)\, dx$$

In addition, formula (4.9) implies that

$$\int_R^\infty \exp(iux)\, P_0(x)\, dx = \int_R^\infty \exp(iux) \left(\frac{\alpha_{jk}^{+}}{x^2} + O(x^{-5/2})\right) dx$$

Since

$$\int_R^\infty \exp(iux) \frac{dx}{x^2} = -iu \ln |u| + r(u)$$

where the function $r(u)$ is analytic at $u=0$, we get that

$$\begin{aligned}\alpha(\mathbf{t}; R) &= \alpha_{jk}^{+} u^2 \ln |u| + \alpha_0(\mathbf{t}; R) + \alpha_1(\mathbf{t}; R)\\ &= \alpha_{jk}^{+}(\mathbf{t}, \omega_{jk})^2 \ln |(\mathbf{t}, \boldsymbol{\omega}_{jk})| + \alpha_0(\mathbf{t}; R) + \alpha_1(\mathbf{t}; R)\end{aligned} \tag{4.12}$$

where the function $\alpha_0(\mathbf{t}; R)$ is analytic at $\mathbf{t}=0$ and $|\alpha_1(\mathbf{t}; R)| \leqslant \text{const} \cdot |\mathbf{t}|^{5/2}$.

Next, differentiating by parts twice, we get that

$$\begin{aligned}\beta(\mathbf{t}; R) = -(uv) \iint_{\{x \geqslant R\}} &\exp[i(ux+vy)]\\ &\times [P(x, y) - Q(x, y)]\, dx\, dy + \beta_0(\mathbf{t}; R)\end{aligned}$$

where the function $\beta_0(\mathbf{t}; R)$ is analytic at $\mathbf{t}=0$. For the sake of brevity, let us write

$$b(\mathbf{t}; R)=\iint_{\{x\geqslant R\}} \exp[i(ux+vy)]\,[P(x,y)-Q(x,y)]\,dx\,dy$$

so that

$$\beta(\mathbf{t}; R)= -uvb(\mathbf{t}; R)+\beta_0(\mathbf{t}; R)$$

Then

$$b(\mathbf{t}; R)-b(0; R)=\iint_{\{x\geqslant R\}} \{\exp[i(ux+vy)]-1\}[P(x,y)-Q(x,y)]\,dx\,dy$$

Let us decompose the last integral into the sum of two integrals, one over the domain $\{|\mathbf{t}|^{-1/2}\geqslant x\geqslant R\}$ and the second over the domain $\{x\geqslant |\mathbf{t}|^{-1/2}\}$. In the first domain

$$|\exp[i(ux+vy)]-1|\leqslant 2\,|\mathbf{t}|^{1/2}$$

so

$$\begin{aligned}\Bigg|\iint_{\{|\mathbf{t}|^{-1/2}\geqslant x\geqslant R\}} &\{\exp[i(ux+vy)]-1\}[P(x,y)-Q(x,y)]\,dx\,dy\Bigg|\\ &\leqslant 2\,|\mathbf{t}|^{1/2}\iint_{\{|\mathbf{t}|^{-1/2}\geqslant x\geqslant R\}} |P(x,y)-Q(x,y)|\,dx\,dy\\ &\leqslant C\,|\mathbf{t}|^{1/2}\int_R^{|\mathbf{t}|^{-1/2}}\frac{dx}{x^2}\leqslant C_0\,|\mathbf{t}|^{1/2}\end{aligned}$$

The integral over the second domain is estimated as follows:

$$\begin{aligned}\Bigg|\iint_{\{x\geqslant |\mathbf{t}|^{-1/2}\}} &\{\exp[i(ux+vy)]-1)]-1\}[P(x,y)-Q(x,y)]\,dx\,dy\Bigg|\\ &\leqslant C\int_{|\mathbf{t}|^{-1/2}}^{\infty}\frac{dx}{x^2}\leqslant C\,|\mathbf{t}|^{1/2}\end{aligned}$$

Thus we get the estimate

$$|b(\mathbf{t}; R)-b(0; \mathbf{R})|\leqslant C_1\,|\mathbf{t}|^{1/2}$$

This implies that

$$\beta(\mathbf{t}; R) = \beta_1(\mathbf{t}; R) + \beta_2(\mathbf{t}; R) \tag{4.13}$$

where

$$|\beta_1(\mathbf{t}; R)| \leqslant C_1 \, |\mathbf{t}|^{5/2} \tag{4.14}$$

and $\beta_2(\mathbf{t}; R)$ is analytic at $\mathbf{t} = 0$.

Summarizing relations (4.10)–(4.14), we get that

$$\chi_{jk}^{+}(\mathbf{t}; R) = \alpha_{jk}^{+}(\mathbf{t}, \boldsymbol{\omega}_{jk})^2 \ln |(\mathbf{t}, \boldsymbol{\omega}_{jk})| + \xi(\mathbf{t}; R) + \eta(\mathbf{t}; R)$$

where $\xi(\mathbf{t}; R)$ satisfies the estimate $|\xi(\mathbf{t}; R)| \leqslant \text{const} \cdot |\mathbf{t}|^{5/2}$ and $\eta(\mathbf{t}; R)$ is analytic at $\mathbf{t} = 0$. A similar formula is valid for $\chi_{jk}^{-}(\mathbf{t}; R)$. Summing up all these formulas in j, k, and $\pm$, we get that

$$\chi(\mathbf{t}) = \sum_{j=1}^{N} \sum_{k=1}^{N_j} \alpha_{jk}(\mathbf{t}, \boldsymbol{\omega}_{jk})^2 \ln |(\mathbf{t}, \boldsymbol{\omega}_{jk})| + \theta(\mathbf{t}) + \zeta(\mathbf{t})$$

where

$$|\theta(\mathbf{t})| \leqslant \text{const} \cdot |\mathbf{t}|^{5/2}$$

and $\zeta(\mathbf{t})$ is analytic at $\mathbf{t} = 0$.

Proposition 4.1 implies that

$$\frac{\partial \chi}{\partial t_1}(0) = \frac{\partial \chi}{\partial t_2}(0) = 0$$

In addition, $\chi(0) = 1$, so

$$\zeta(0) = 1, \qquad \frac{\partial \zeta}{\partial t_1}(0) = \frac{\partial \zeta}{\partial t_2}(0) = 0$$

Hence

$$\ln \chi(\mathbf{t}) = \sum_{j=1}^{N} \sum_{k=1}^{N_j} \alpha_{jk}(\mathbf{t}, \boldsymbol{\omega}_{jk})^2 \ln |(\mathbf{t}, \boldsymbol{\omega}_{jk})| + \theta_0(\mathbf{t}) + \zeta_0(\mathbf{t})$$

where

$$|\theta_0(\mathbf{t})| \leqslant \text{const} \cdot |\mathbf{t}|^{5/2}$$

and $\zeta_0(\mathbf{t})$ is analytic at $\mathbf{t} = 0$ and

$$\zeta_0(0) = \frac{\partial \zeta_0}{\partial t_1}(0) = \frac{\partial \zeta_0}{\partial t_2}(0) = 0$$

This implies that

$$\zeta_0(\mathbf{t}) = (A\mathbf{t}, \mathbf{t}) + O(|\mathbf{t}|^3)$$

and so

$$\ln \chi(\mathbf{t}) = \sum_{j=1}^{N} \sum_{k=1}^{N_j} \alpha_{jk}(\mathbf{t}, \boldsymbol{\omega}_{jk})^2 \ln |(\mathbf{t}, \boldsymbol{\omega}_{jk})| + (A\mathbf{t}, \mathbf{t}) + O(|\mathbf{t}|^{5/2})$$

Theorem 4.4 is proved.

5. ESTIMATES OF CORRELATIONS

In the previous section we have shown that $\langle |\mathbf{r}(\lambda)|^2 \rangle = \infty$. Now we will show that

$$\langle |\mathbf{r}(\lambda)| \, |\mathbf{r}(T^n \lambda)| \rangle < \infty \tag{5.1}$$

for any $n \neq 0$.

Denote by $v_n(d\mathbf{r}_1 \, d\mathbf{r}_2)$ the joint distribution of vectors $\mathbf{r}(\lambda)$, $\mathbf{r}(T^n \lambda)$ with respect to the Liouville measure $\mu_0(d\lambda)$.

Proposition 5.1. The distribution $v_n(d\mathbf{r}_1 \, d\mathbf{r}_2)$ is invariant with respect to the transformation P: $(\mathbf{r}_1, \mathbf{r}_2) \to -(\mathbf{r}_2, \mathbf{r}_1)$.

Proof. In this proof we will denote by $(\mathbf{r}_1, \mathbf{r}_2)$ the pair of vectors $\mathbf{r}_1, \mathbf{r}_2$ and not their scalar product. In addition, $x \overset{d}{=} y$ denotes the equality of distributions of random variables x, y. We have

$$(\mathbf{r}(\lambda), \mathbf{r}(T^n \lambda)) \overset{d}{=} (\mathbf{r}(T^{-n-1} S\lambda), \mathbf{r}(T^{-1} S\lambda))$$

because both T and S preserve μ_0. Next,

$$T^{-n-1} = (STS)^{n+1} = ST^{n+1}S$$

and

$$\mathbf{r}(ST\lambda) \overset{d}{=} -\mathbf{r}(\lambda)$$

so

$$(\mathbf{r}(T^{-n-1} S\lambda), \mathbf{r}(T^{-1} S\lambda)) = (\mathbf{r}(ST^{n+1}\lambda), \mathbf{r}(ST\lambda)) \overset{d}{=} -(\mathbf{r}(T^n \lambda), \mathbf{r}(\lambda))$$

so

$$(\mathbf{r}(\lambda), \mathbf{r}(T^n \lambda)) \overset{d}{=} -(\mathbf{r}(T^n \lambda), \mathbf{r}(\lambda))$$

The proposition is proved.

Corollary. $(|\mathbf{r}(\lambda)|, |\mathbf{r}(T^n\lambda)|) \stackrel{d}{=} (|\mathbf{r}(T^n\lambda)|, |\mathbf{r}(\lambda)|)$.

Now we prove (5.1) for $n=1$. To that end, we calculate the asymptotics of the tail of the distribution $\nu_1(d\mathbf{r}_1, \mathbf{r}_2)$ of vectors $\mathbf{r}(\lambda), \mathbf{r}(T\lambda)$. Assume that $|\mathbf{r}(\lambda)|, |\mathbf{r}(T\lambda)|$ are big enough. Then the vector $[\mathbf{x}(\lambda), \mathbf{x}(T\lambda)]$ belongs to some corridor C_{jk} which means that from $\mathbf{x}(\lambda)$ to $\mathbf{x}(T\lambda)$ the particle moves long inside C_{jk} almost parallel or antiparallel to its direction $\boldsymbol{\omega}_{jk}$. This implies that in the collision at $\mathbf{x}(T\lambda)$ the particle changes very slightly the direction of its velocity and so the next segment of free motion $[\mathbf{x}(T\lambda), \mathbf{x}(T^2\lambda)]$ also lies almost entirely in the corridor C_{jk}. The main observation is that in a typical situation, however,

$$|\mathbf{x}(T^2\lambda) - \mathbf{x}(T\lambda)| \sim \text{const} \cdot [|\mathbf{x}(T\lambda) - \mathbf{x}(\lambda)|^{1/2}] \tag{5.2}$$

(see also refs. 24 and 29). Since the function $\Pr\{|\mathbf{r}(\lambda)| \geqslant R\}$ decreases as $\text{const} \cdot R^{-2}$ when $R \to \infty$, (5.2) implies that $\langle |\mathbf{r}(\lambda)|\, |\mathbf{r}(T\lambda)| \rangle$ is finite. Unfortunately, the constant in (5.2) is not uniform in λ, so we actually need more refined arguments. In what follows we estimate the tail of the distribution $\nu_1(d\mathbf{r}_1, d\mathbf{r}_2)$.

Consider for positive $R_1, \Delta_1, R_2, \Delta_2$ the region in the phase space

$$\begin{aligned} V_{jk}^{\pm}&(R_1, \Delta_1, R_2, \Delta_2) \\ &= \{\lambda \in \Lambda_0 \mid \mathbf{x}(\lambda) \in \partial\Omega_j, [\mathbf{x}(\lambda), \mathbf{x}(T\lambda)] \in C_{jk}, \\ &\quad \pm(\mathbf{r}(\lambda), \boldsymbol{\omega}_{jk}) > 0, R_1 < |\mathbf{r}(\lambda)| < R_1 + \Delta_1, R_2 < |\mathbf{r}(T\lambda)| < R_2 + \Delta_2\} \end{aligned} \tag{5.3}$$

We will estimate the probability

$$p_{jk}^{\pm}(R_1, \Delta_1, R_2, \Delta_2) = \int_{V_{jk}^{\pm}(R_1, \Delta_1, R_2, \Delta_2)} \mu_0(d\lambda)$$

of such types of regions. Consider a big integer number n_0 such that any segment of free motion of the length greater than $(n_0)^2$ belongs to some corridor and put for $m, n \geqslant n_0$,

$$q_{jk}^{\pm}(m, n) = p_{jk}^{\pm}(m^2, 2m+1, n^2, 2n+1)$$

Proposition 5.2. The numbers $q_{jk}^{\pm}(m, n) \geqslant 0$ have the following properties:

(i) $q_{jk}^{\pm}(n, m) = q_{jk}^{\pm}(m, n)$.

(ii) $\exists C, C_0 > 0$ such that

$$q_{jk}^{\pm}(m, n) \begin{cases} = 0 & \text{if} \quad n < C\sqrt{m} \\ \leqslant C_0 \dfrac{R_m + R_n}{(R_m R_n)^3} \Delta_m \Delta_n & \text{if} \quad C\sqrt{m} \leqslant n \leqslant m \end{cases}$$

where $R_m = m^2$, $\Delta_m = 2m + 1$.

Proof. (i) Follows from the Corollary to Proposition 5.1. (ii) Let us introduce some notations. Let $\mathbf{x}(\lambda) \in \partial\Omega_j$, $[\mathbf{x}(\lambda), \mathbf{x}(T\lambda)] \in C_{jk}$, $\mathbf{x}(T\lambda) \in \partial\Omega_\gamma$, and $\mathbf{x}_j = \partial\Omega_j \cap C_{jk}$, $\mathbf{x}_\gamma = \partial\Omega_\gamma \cap C_{jk}$. Consider the coordinate system (x, y) with the orthonormal basis $\mathbf{f}_1 = \boldsymbol{\omega}_{jk}$, $\mathbf{f}_2$ and with the origin at the point $\mathbf{x}_\gamma$, so that $\mathbf{x}$ is the coordinate along the corridor C_{jk}, y is the orthogonal one, and $\mathbf{x}_\gamma = (0, 0)$. For the sake of definiteness we will assume that the scatterer Ω_γ lies in the upper half-plane $\{y \geqslant 0\}$ and the x coordinate of the vector $\mathbf{r}(\lambda) = \mathbf{x}(T\lambda) - \mathbf{x}(\lambda)$ is positive.

Consider the point $\mathbf{z}(\lambda) = (z(\lambda), 0)$ of the intersection of the segment $[\mathbf{x}(\lambda), \mathbf{x}(T\lambda)]$ with the x axis. It is clear that

$$|z(\lambda)| \leqslant h \tag{5.4}$$

where h is the period of the scatterer lattice in the direction $\boldsymbol{\omega}_{jk}$, so $|z(\lambda)|$ is uniformly bounded. The same calculations as we used in the proof of Proposition 4.2 above lead to the following asymptotic formula for the density $p(z)$ of the distribution of $z(\lambda)$ with respect to the Liouville measure $\mu_0(d\lambda)$:

$$p(z) = \frac{2\alpha_{jk}^{+}}{R^3} + O(R^{-7/2}), \qquad R \to \infty, \quad |z(\lambda)| \leqslant h \tag{5.5}$$

where $R = |\mathbf{x}_\gamma - \mathbf{x}_j|$. Remark that $|\mathbf{x} - \mathbf{y}| = R + O(1)$ for any $\mathbf{x} \in \partial\Omega_j$, $\mathbf{y} \in \partial\Omega_\gamma$, so (5.5) remains valid if we change R for $|\mathbf{x} - \mathbf{y}|$, where $\mathbf{x}$, $\mathbf{y}$ are arbitrary points at $\partial\Omega_j$, $\partial\Omega_\gamma$, respectively.

Let us estimate now $R' = |\mathbf{r}(T\lambda)| = |\mathbf{x}(T^2\lambda) - \mathbf{x}(T\lambda)|$. Denote by ζ, ζ' the angles (without sign) between the x axis and the vectors $\mathbf{r}(\lambda)$, $\mathbf{r}(T\lambda)$, respectively. Consider the line l which is parallel to $[\mathbf{x}(\lambda), \mathbf{x}(T\lambda)]$ and is tangent to Ω_γ from below. Let $\mathbf{v} = (v, 0)$ be the intersection point of l and the x axis. Then direct geometrical calculations give the formula

$$\zeta' = -\zeta + 2\left[\frac{2\zeta}{\rho}(v - z)\right]^{1/2} \{1 + O([\zeta(v - z)]^{1/2})\}$$

where ρ is the curvature radius of $\partial\Omega_\gamma$ at $\mathbf{x}_\gamma$. Simple further calculations lead to the asymptotic formula

$$R' = \frac{R}{-1+2[1-(2z/\rho d)\,R]^{1/2}} + O(1) \tag{5.6}$$

Let us discuss this formula. One can see that in the typical case $-z$ is of the order of 1, and in that case R' is of the order of $\sqrt{R}$. However, for small z of the order of R^{-1}, R' is of the order of R. In particular, $R' = R + O(1)$ for $z = 0$, which corresponds to the reflection at $\mathbf{x}_\gamma$. We are interested in the case when $\mathbf{R}' \leqslant R + \text{const}\cdot\sqrt{R}$, so we may assume that

$$z = z(\lambda) \leqslant R^{-4/3}$$

Due to (5.6), the condition (5.4) implies the inequality

$$R' \geqslant \text{const}\cdot\sqrt{R} \tag{5.7}$$

Assume now that $R_m \leqslant R \leqslant R_m + \Delta_m$, $R_m = m^2$, $\Delta_m = 2m+1$. Consider such z in (5.6) that $R_n \leqslant R' \leqslant R_n + \Delta_n$, $n \leqslant m$. Direct calculations based on (5.6) show that the length of the segment of those z's is equal to

$$\Delta z = \frac{R_m + R_n}{(R_n)^3}\,\frac{\rho d}{4}\,\Delta_n[1 + O(n^{-1})]$$

so according to (5.5) the probability of this segment does not exceed

$$\text{const}\,\frac{R_m + R_n}{(R_m R_n)^3}\,\Delta_n$$

This gives us the estimate of the probability of the set $V_{jk}^{\pm}(R_m, \Delta_m, R_n, \Delta_n) \cap \{\mathbf{x}(T\lambda) \in \partial\Omega_\gamma\}$. Summing up over all admissible γ, we get the estimate

$$q_{jk}^{\pm}(m,n) = \Pr\{V_{jk}^{\pm}(R_m, \Delta_m, R_n, \Delta_n)\} \leqslant \text{const}\cdot\frac{R_m + R_n}{(R_m R_n)^3}\,\Delta_n \Delta_m$$

The inequality (5.7) implies that $q_{jk}^{\pm}(m,n) = 0$ if $n < \text{const}\cdot\sqrt{m}$. Proposition 5.2 is proved.

Proposition 5.3. $\langle |\mathbf{r}(\lambda|\ |\mathbf{r}(T\lambda)|\rangle < \infty$.

Proof. Write

$$V(R) = \{\lambda \in \Lambda_0 \mid |\mathbf{r}(\lambda)| \geqslant R,\ |\mathbf{r}(T\lambda)| \geqslant R\}$$

Since

$$\int_{\Lambda_0 \setminus V(R)} |\mathbf{r}(\lambda)|\ |\mathbf{r}(T\lambda)|\ \mu_0(d\lambda) \leqslant 2R \int_{\Lambda_0 \setminus V(R)} |\mathbf{r}(\lambda)|\ \mu_0(d\lambda) \leqslant 2R \langle |\mathbf{r}(\lambda)| \rangle < \infty$$

it is enough to estimate

$$\int_{V(R)} |\mathbf{r}(\lambda)|\ |\mathbf{r}(T\lambda)|\ \mu_0(d\lambda)$$

for some $R > 0$. Moreover, since for large R,

$$\int_{V(R)} |\mathbf{r}(\lambda)|\ |\mathbf{r}(T\lambda)|\ \mu_0(d\lambda) = \sum_{j=1}^{N} \sum_{k=1}^{N_j} \sum_{\pm} \int_{V_{jk}^{\pm}(R,\, \infty,\, R,\, \infty)} |\mathbf{r}(\lambda)|\ |\mathbf{r}(T\lambda)|\ \mu_0(d\lambda)$$

where the sets $V_{jk}^{\pm}(R_1, \Delta_1, R_2, \Delta_2)$ were defined in (5.3), it is enough to estimate

$$\int_{V_{jk}^{\pm}(R, \infty, R, \infty)} |\mathbf{r}(\lambda)|\ |\mathbf{r}(T\lambda)|\ \mu_0(d\lambda)$$

Consider a large integer number n_0 such that any segment of free motion of the length greater than $(n_0)^2$ belongs to some corridor. Then we have

$$\begin{aligned}
&\int_{V_{jk}^{\pm}((n_0)^2,\, \infty,\, (n_0)^2,\, \infty)} |\mathbf{r}(\lambda)|\ |\mathbf{r}(T\lambda)|\ \mu_0(d\lambda) \\
&\quad = \sum_{m=n_0}^{\infty} \sum_{n=n_0}^{\infty} \int_{V_{jk}^{\pm}(m^2, 2m+1, n^2, 2n+1)} |\mathbf{r}(\lambda)|\ |\mathbf{r}(T\lambda)|\ \mu_0(d\lambda) \\
&\quad \leqslant \sum_{m=n_0}^{\infty} \sum_{n=n_0}^{\infty} (m+1)^2\,(n+1)^2 \int_{V_{jk}^{\pm}(m^2, 2m+1, n^2, 2n+1)} \mu_0(d\lambda) \\
&\quad = \sum_{m=n_0}^{\infty} \sum_{n=n_0}^{\infty} (m+1)^2\,(n+1)^2\, q_{jk}^{\pm}(m, n)
\end{aligned}$$

By Proposition 5.2

$$\begin{aligned}
&\sum_{m=n_0}^{\infty} \sum_{n=n_0}^{\infty} (m+1)^2\,(n+1)^2\, q_{jk}^{\pm}(m, n) \\
&\quad \leqslant \sum_{m=n_0}^{\infty} \sum_{n=[C\sqrt{m}]}^{m} (m+1)^2\,(n+1)^2\, C_0 \frac{m^2+n^2}{m^6 n^6} (2m+1)(2n+1) \\
&\quad \leqslant C_1 \sum_{m=n_0}^{\infty} m^{-1} \sum_{n=[C\sqrt{m}]}^{m} \frac{1}{n^3} \leqslant C_2 \sum_{m=n_0}^{\infty} m^{-2} < \infty
\end{aligned}$$

Proposition 5.3 is proved.

Now we turn to the proof of the relation (5.1) for any $n \geqslant 1$. Consider a trajectory of the particle in a corridor C_{jk}. Let

$$\mathbf{x}(\lambda) \in \partial\Omega_j, \quad \mathbf{x}(T\lambda) \in \partial\Omega_{\gamma_1}, \quad \mathbf{x}(T^2\lambda) \in \partial\Omega_{\gamma_2},..., \mathbf{x}(T^{n+1}\lambda) \in \partial\Omega_{\gamma_{n+1}}$$

where all the scatterers Ω_j, $\Omega_{\gamma_1},..., \Omega_{\gamma_{n+1}}$ touch the corridor C_{jk}. Introduce a coordinate system with the orthonormal basis $\mathbf{f}_1 = \boldsymbol{\omega}_{jk}$, $\mathbf{f}_2$ and with the origin at $\mathbf{x}_{jk} = \partial\Omega_j \cap C_{jk}$. For the sake of definiteness we will assume that the particle moves to the right, so that if $\mathbf{x}(T^m\lambda) = (x(T^m\lambda), y(T^m\lambda))$, then $x(T^m\lambda) > x(T^{m-1}\lambda)$ for $m = 1,..., n$. In addition, we will assume that the even scatterers $\partial\Omega_j \equiv \partial\Omega_{\gamma_0}$, $\partial\Omega_{\gamma_2}$, $\partial\Omega_{\gamma_4}$,... lie in the lower half-plane $\{y \leqslant 0\}$, while the odd ones $\partial\Omega_{\gamma_1}$, $\partial\Omega_{\gamma_3}$,... lie in the upper half-plane $\{y \geqslant d_{jk}\}$. In principle it is possible that two subsequent scatterers $\partial\Omega_{\gamma_{m-1}}$, $\partial\Omega_{\gamma_m}$ lie on the same side of the corridor C_{jk} (see Fig. 3). We shall consider this case later.

Denote by $\mathbf{z}_m(\lambda) = (z_m(\lambda), u_m(\lambda))$ the intersection point of the segment $[\mathbf{x}(T^{m-1}\lambda), \mathbf{x}(T^m\lambda)]$ with ∂C_{jk}, which lies near the point $\mathbf{x}(T^m\lambda)$. It is noteworthy that $u_m(\lambda) = 0$ for even m and $= d_{jk}$ for odd m. Let us fix the initial point $\mathbf{x}(\lambda)$ and consider small perturbations of the velocity vector at the initial moment. In other words, we consider $\lambda = (j, s, \eta)$, where j, s are fixed and η is varied. We want to study the dependence of $z_m(\lambda)$ on η. Since j, s are fixed, we redenote $z_m(\lambda)$ by $z_m(\eta)$. Write $\tau_m = |\mathbf{x}(T^m\lambda) - \mathbf{x}(T^{m-1}\lambda)|$ and call ρ_m the curvature radius of $\partial\Omega_{\gamma m}$ at $\mathbf{x}(T^m\lambda)$.

Proposition 5.4. We have

(i) $\dfrac{dz_m}{d\eta} > 0, \qquad m \geqslant 1$

(ii) $$\frac{dz_m}{dz_{m-1}} = \frac{4(\tau_m)^3}{\rho_{m-1}(\tau_m + \tau_{m-1})} \times [1 + O((\tau_m)^{-1} + (\tau_{m-1})^{-1})], \qquad m \geqslant 2$$

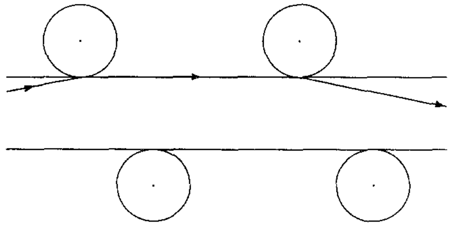

Fig. 3. Subsequent reflections from neighboring scatterers.

Remark. Simple geometrical considerations show that $\tau_{m-1} \leqslant \text{const} \cdot \tau_m^2$, so that point (ii) implies that $dz_m/dz_{m-1} \geqslant \text{const} \cdot \tau_m$, which means that the map $z_{m-1} \to z_m$ is strongly expanding.

Proof. Denote by $\zeta_m = \zeta_m(\eta)$ the angle without sign between $\mathbf{r}(T^{m-1}\lambda) = \mathbf{x}(T^m\lambda) - \mathbf{x}(T^{m-1}\lambda)$ and $\boldsymbol{\omega}_{jk}$. We will prove (i), (ii) by induction. We have

$$\zeta_1 = \frac{\pi}{2} - \alpha - \eta$$

where α is the angle between the tangent line to $\partial\Omega_j$ at $\mathbf{x}(\lambda)$ and $\boldsymbol{\omega}_{jk}$, and

$$\frac{d-y}{z_1 - x} = \tan \zeta_1 \tag{5.8}$$

where $\mathbf{x}(\lambda) = (x, y)$, $d = d_{jk}$. So

$$\frac{d\zeta_1}{d\eta} = -1 \tag{5.9}$$

and

$$\frac{dz_1}{d\eta} = -\frac{dz_1}{d\zeta_1} = (d-y)\left[1 + \left(\frac{z_1 - x}{d-y}\right)^2\right] \tag{5.10}$$

Let us assume now that

$$\frac{d\zeta_{m-1}}{d\eta} < 0, \qquad \frac{dz_{m-1}}{d\eta} > 0 \tag{5.11}$$

for some $m \geqslant 2$ and prove similar inequalities for $d\zeta_m/d\eta$, $dz_m/d\eta$. For the sake of definiteness we will assume that m is even.

Consider a small perturbation (z, ζ) of the pair (z_{m-1}, ζ_{m-1}) and a trajectory starting at the point (z, d) with the velocity $\mathbf{v} = (\cos\zeta, \sin\zeta)$. Let $\mathbf{v} = (\cos\zeta', -\sin\zeta')$ be the velocity in this trajectory after the reflection from $\Omega_{\gamma_{m-1}}$ and let $(z', 0)$ be the intersection point of the trajectory with the x axis after that reflection. We have a map

$$S_{m-1}\colon \quad (z, \zeta) \to (z', \zeta')$$

which is defined in a neighborhood of the point (z_{m-1}, ζ_{m-1}). By construction,

$$S_{m-1}\colon \quad (z_{m-1}, \zeta_{m-1}) \to (z_m, \zeta_m)$$

Direct calculations give that

$$\frac{\partial z'}{\partial z} = \frac{\sin \zeta}{\sin \zeta'} + \frac{\sin \zeta}{\sin \zeta'} \cdot \frac{2t'}{\rho \cos \varphi} \tag{5.12}$$

where t' is the length of the segment between the reflection point at $\partial \Omega_{\gamma_{m-1}}$ and $\mathbf{z}' = (z', 0)$, ρ is the curvature radius of $\partial \Omega_{\gamma_{m-1}}$ at the reflection point, and φ is the angle of the reflection. Similarly,

$$\begin{aligned} -\frac{\partial z'}{\partial \zeta} &= \frac{t + t'}{\sin \zeta'} - \frac{2tt'}{\rho \sin \zeta' \cos \varphi} \\ -\frac{\partial \zeta'}{\partial z} &= \frac{2 \sin \zeta}{\rho \cos \varphi} \\ \frac{\partial \zeta'}{\partial \zeta} &= 1 + \frac{2t}{\rho \cos \varphi} \end{aligned} \tag{5.13}$$

where t is the length of the segment between the point (z, d) and the reflection point at $\partial \Omega_{\gamma_{m-1}}$. We have

$$\frac{dz_m}{d\eta} = \frac{\partial z'}{\partial z}(z_{m-1}, \zeta_{m-1}) \frac{dz_{m-1}}{d\eta} + \frac{\partial z'}{\partial \zeta}(z_{m-1}, \zeta_{m-1}) \frac{d\zeta_{m-1}}{d\eta}$$

Due to (5.12), (5.13),

$$\frac{\partial z'}{\partial z}(z_{m-1}, \zeta_{m-1}) > 0, \qquad -\frac{\partial z'}{\partial \zeta}(z_{m-1}, \zeta_{m-1}) > 0$$

so in view of (5.11)

$$\frac{dz_m}{d\eta} > 0$$

Similarly,

$$\frac{d\zeta_m}{d\eta} = \frac{\partial \zeta'}{\partial z}(z_{m-1}, \zeta_{m-1}) \frac{dz_{m-1}}{d\eta} + \frac{\partial \zeta'}{\partial \zeta}(z_{m-1}, \zeta_{m-1}) \frac{d\zeta_{m-1}}{d\eta} < 0$$

Part (i) is proved.

By (5.11) $d\zeta_{m-1}/dz_{m-1} < 0$. Equations (5.12), (5.13) imply the recursive "continued fraction" formula[20,22]

$$-\frac{d\zeta_m}{dz_m} = \cfrac{\sin \zeta_m}{t_m + \cfrac{1}{\cfrac{2}{\rho_{m-1} \cos \eta_{m-1}} + \cfrac{1}{t'_{m-1} - \cfrac{\sin \zeta_{m-1}}{d\zeta_{m-1}/dz_{m-1}}}}}$$

where

$$t_m = |\mathbf{x}(T^{m-1}\lambda) - \mathbf{z}_m(\lambda)|, \qquad t'_{m-1} = |\mathbf{x}(T^{m-1}\lambda) - \mathbf{z}_{m-1}(\lambda)|$$

and η_{m-1} is the reflection angle at the point $\mathbf{x}(T^{m-1}\lambda)$. Since $\sin\zeta_m = O(1/t_m)$ we get that

$$\left|\frac{d\zeta_m}{dz_m}\right| = O\left(\frac{1}{t_m^2}\right) \tag{5.14}$$

Next,

$$\frac{dz_m}{dz_{m-1}} = \frac{\partial z'}{\partial z}(z_{m-1}, \zeta_{m-1}) + \frac{\partial z'}{\partial \zeta}(z_{m-1}, \zeta_{m-1})\frac{d\zeta_{m-1}}{dz_{m-1}} \tag{5.15}$$

By (5.12)

$$\frac{\partial z'}{\partial z}(z_{m-1}, \zeta_{m-1}) = \frac{\sin\zeta_{m-1}}{\sin\zeta_m} + \frac{\sin\zeta_{m-1}}{\sin\zeta_m}\cdot\frac{2t_m}{\rho_{m-1}\cos\eta_{m-1}}$$

Remark that $2\eta_{m-1} + \zeta_m = \pi$, so

$$\cos\eta_{m-1} = \sin\frac{\zeta_{m-1} + \zeta_m}{2}$$

and

$$\frac{\partial z'}{\partial z}(z_{m-1}, \zeta_{m-1}) = \frac{\sin\zeta_{m-1}}{\sin\zeta_m} + \frac{\sin\zeta_{m-1}}{\sin\zeta_m}\cdot\frac{2t_m}{\rho_{m-1}\sin\frac{1}{2}(\zeta_{m-1} + \zeta_m)}$$

In addition, $t_m \equiv |\mathbf{x}(T^{m-1}\lambda) - \mathbf{z}m(\lambda)| = \tau_m + O(1)$ and

$$\sin\zeta_m = \frac{1}{\tau_m} + O\left(\frac{1}{\tau_m^2}\right)$$

so

$$\begin{aligned}\frac{\partial z'}{\partial z}(z_{m-1}, \zeta_{m-1}) &= \frac{2(\tau_m)^2}{\rho_{m-1}\tau_{m-1}(\tau_{m-1}^{-1} + \tau_m^{-1})/2}[1 + O(\tau_{m-1}^{-1} + \tau_m^{-1})] \\ &= \frac{4(\tau_m)^3}{\rho_{m-1}(\tau_{m-1} + \tau_m)}[1 + O(\tau_{m-1}^{-1} + \tau_m^{-1})]\end{aligned} \tag{5.16}$$

By (5.13)

$$\frac{\partial z'}{\partial \zeta}(z_{m-1}, \zeta_{m-1}) = \frac{t_m + t'_{m-1}}{\sin\zeta_m} + \frac{2t_m t'_{m-1}}{\rho_{m-1}\sin\zeta_m\sin\frac{1}{2}(\zeta_{m-1} + \zeta_m)}$$

Since $t'_{m-1} = |\mathbf{x}(T^{m-1}\lambda) - \mathbf{z}_{m-1}(\lambda)| = O(1)$, we get that

$$\left|\frac{\partial z'}{\partial \zeta}(z_{m-1}, \zeta_{m-1})\right| = O\left(t_m^2\left(\frac{1}{t_m}+\frac{1}{t_{m-1}}\right)^{-1}\right)$$

so in view of (5.14)

$$\begin{aligned}\left|\frac{\partial z'}{\partial \zeta}(z_{m-1}, \zeta_{m-1})\frac{d\zeta_{m-1}}{dz_{m-1}}\right| &\leqslant C\left(\frac{t_m^2}{t_{m-1}^2}\left(\frac{1}{t_m}+\frac{1}{t_{m-1}}\right)^{-1}\right)\\ &= C\frac{t_m^3}{t_{m-1}(t_{m-1}+t_m)}\\ &\leqslant C_0\frac{\tau_m^3}{\tau_{m-1}+\tau_m}\cdot\tau_{m-1}^{-1}\end{aligned}$$

Hence, due to (5.15), (5.16) we get that

$$\frac{dz_m}{dz_{m-1}} = \frac{4(\tau_m)^3}{\rho_{m-1}(\tau_{m-1}+\tau_m)}[1+O(\tau_{m-1}^{-1}+\tau_m^{-1}]$$

Proposition 5.4 is proved.

Consider now a trajectory of the particle in a corridor C_{jk},

$$\mathbf{x}(\lambda)\in\partial\Omega_j,\ \mathbf{x}(T\lambda)\in\partial\Omega_{\gamma_1},\ \mathbf{x}(T^2\lambda)\in\partial\Omega_{\gamma_2},...,\ \mathbf{x}(T^{n+1}\lambda)\in\partial\Omega_{\gamma_{n+1}}$$

where all the scatterers $\Omega_j, \Omega_{\gamma_1},..., \Omega_{\gamma_{n+1}}$ touch the corridor C_{jk} and the reflections represented in Fig. 3 are admitted.

Proposition 5.5. We have

(i) $\dfrac{dz_m}{d\eta} > 0,\ m \geqslant 1$

(ii) $$\frac{dz_m}{dz_{m-1}} = \frac{4(\tau_m)^3}{\rho_{m-1}(\tau_m+\tau_{m-1})} \times [1+O((\tau_m)^{-1}+(\tau_{m-1})^{-1})], \qquad m\geqslant 2$$

if the scatterers $\Omega_{\gamma_{m-2}}$, Ω_{γ_m} lie on another side from the corridor C_{jk} than $\Omega_{\gamma_{m-1}}$ does.

(iii) $$\frac{dz_m}{dz_{m-2}} = \frac{4\tau_{m-1}(\tau_m)^2}{\rho_{m-1}\rho_{m-2}\tau_{m-2}\cos\eta_{m-2}\cos\eta_{m-1}} \times [1+O((\tau_m)^{-1}+(\tau_{m-2})^{-1})], \qquad m\geqslant 3$$

if the scatterers $\Omega_{\gamma_{m-3}}$, Ω_{γ_m} lie on another side from the corridor C_{jk} than $\Omega_{\gamma_{m-2}}$, $\Omega_{\gamma_{m-1}}$ do.

Proof of this proposition goes similarly to the proof of Proposition 5.4 and so we omit it. Let us turn now to the proof of the main result of the present section.

Theorem 5.6. For any $n \geqslant 1$, $\langle |\mathbf{r}(\lambda)|\, |\mathbf{r}(T^n\lambda)| \rangle < \infty$.

Proof. Denote

$$V(R) = \{\lambda \in \Lambda_0 |\ |\mathbf{r}(\lambda)| \geqslant R,\ |\mathbf{r}(T^n\lambda)| \geqslant R\}$$

Since

$$\int_{\Lambda_0 \setminus V(R)} |\mathbf{r}(\lambda)|\, |\mathbf{r}(T^n\lambda)|\, \mu_0(d\lambda) \leqslant 2R \int_{\Lambda_0 \setminus V(R)} |\mathbf{r}(\lambda)|\, \mu_0(d\lambda)$$
$$\leqslant 2R\langle |\mathbf{r}(\lambda)| \rangle < \infty$$

it is enough to estimate

$$\int_{V(R)} |\mathbf{r}(\lambda)|\, |\mathbf{r}(T^n\lambda)|\, \mu_0(d\lambda)$$

for some $R > 0$. If R is big enough and $|\mathbf{r}(\lambda)| > R$, then the whole trajectory $\mathbf{x}(\lambda)$, $\mathbf{x}(T\lambda)$,..., $\mathbf{x}(T^{n+1}\lambda)$ belongs to a corridor C_{jk} in the sense that

$$\mathbf{x}(\lambda) \in \partial\Omega_j, \quad \mathbf{x}(T\lambda) \in \partial\Omega_{\gamma_1}, \quad \mathbf{x}(T^2\lambda) \in \partial\Omega_{\gamma_2},...,$$
$$\mathbf{x}(T^{n+1}\lambda) \in \partial\Omega_{\gamma_{n+1}} \tag{5.17}$$

where the scatterers $\Omega_j, \Omega_{\gamma_1},..., \Omega_{\gamma_{n+1}}$ touch the corridor C_{jk}. Denote by $V_{jk}^{\pm}(R)$ the set of $\lambda \in V(R)$ for which the trajectory $\mathbf{x}(\lambda)$, $\mathbf{x}(T\lambda)$,..., $\mathbf{x}(T^{n+1}\lambda)$ belongs to the corridor C_{jk} and goes in the direction $\pm\boldsymbol{\omega}_{jk}$. We will estimate

$$\int_{V_{jk}^{+}} |\mathbf{r}(\lambda)|\, |\mathbf{r}(T^n\lambda)|\, \mu_0(d\lambda)$$

(we chose the sign + for the sake of definiteness). Let us fix scatterers Ω_{γ_1}, Ω_{γ_2},..., $\Omega_{\gamma_{n+1}}$ and denote by $V_{jk}^{+}(R; \gamma_1, \gamma_2,..., \gamma_{n+1})$ the set of $\lambda \in V_{jk}^{+}(R)$ for which (5.17) holds. Actually the set $V_{jk}^{+}(R; \gamma_1, \gamma_2,..., \gamma_{n+1})$ can be empty and we will assume that this is not the case. We will also assume at first that the even scatterers $\partial\Omega_j \equiv \partial\Omega_{\gamma_0}$, $\partial\Omega_{\gamma_2}$, $\partial\Omega_{\gamma_4}$,..., lie in the lower half-plane

$\{y \leqslant 0\}$, while the odd ones $\partial\Omega_{\gamma_1}$, $\partial\Omega_{\gamma_3},\dots$, lie in the upper half-plane $\{y \geqslant d_{jk}\}$. Accordingly, we write

$$G_n^{(0)} = \{\Gamma = \{\gamma_1, \gamma_2,\dots, \gamma_{n+1}\} \mid \Omega_{\gamma_0} \equiv \Omega_j, \Omega_{\gamma_2},\dots, \subset \{y \leqslant 0\};$$
$$\Omega_{\gamma_1}, \Omega_{\gamma_3},\dots, \subset \{y \geqslant d_{jk}\};$$
$$\exists \lambda \in \Lambda_0 : \mathbf{x}(T^m \lambda) \in \partial\Omega_{\gamma_m},\ m = 0, 1,\dots, n+1\}$$

Let us estimate

$$\int_{V_{jk}^{+}(R;\gamma_1,\gamma_2,\dots,\gamma_{n+1})} |\mathbf{r}(\lambda)|\ |\mathbf{r}(T^n\lambda)|\ \mu_0(d\lambda)$$

for $\{\gamma_1,\dots, \gamma_{n+1}\} \in G_n^{(0)}$. Let

$$t_m = |\mathbf{x}_{\gamma_{m+1}} - \mathbf{x}_{\gamma_m}|, \qquad m = 0,\dots, n$$

where

$$\mathbf{x}_{\gamma_m} = \partial\Omega_{\gamma_m} \cap C_{jk}, \qquad m = 0,\dots, n+1$$

Since $\mathbf{x}(T^m\lambda) \in \partial\Omega_{\gamma_m}$, we have that

$$|\mathbf{r}(T^m\lambda)| = t_m + o(1), \qquad m = 0,\dots, n, \quad R \to \infty$$

so

$$\int_{V_{jk}^{+}(R;\gamma_1,\gamma_2,\dots,\gamma_{n+1})} |\mathbf{r}(\lambda)|\ |\mathbf{r}(T^n\lambda)|\ \mu_0(d\lambda)$$
$$= t_0 t_n \int_{V_{jk}^{+}(R;\gamma_1,\gamma_2,\dots,\gamma_{n+1})} \mu_0(d\lambda)[1 + o(1)]$$
$$= t_0 t_n \Pr\{V_{jk}^{+}(R;\gamma_1,\gamma_2,\dots,\gamma_{n+1})\}[1 + o(1)] \tag{5.18}$$

To estimate $\Pr\{V_{jk}^{+}(R;\gamma_1,\gamma_2,\dots,\gamma_{n+1})\}$, we fix in addition some point $\mathbf{x} \in \partial\Omega_j$ with a natural coordinate s, which lies near the point $\mathbf{x}_{jk} = \partial\Omega_j \cap C_{jk}$, and consider the set

$$\Delta_s = \{\eta \mid \lambda = (j, s, \eta) \in V_{jk}^{+}(R;\gamma_1,\gamma_2,\dots,\gamma_{n+1})\}$$

so that

$$\Pr\{V_{jk}^{+}(R;\gamma_1,\gamma_2,\dots,\gamma_{n+1})\} = Z^{-1} \int ds \int_{\Delta_s} \cos\eta\ d\eta$$

where

$$Z = 2 \sum_{j=1}^{N} |\partial \Omega_j|$$

For the sake of brevity we do not indicate the dependence of Δ_s on j, k, R, $\gamma_1, \gamma_2, \ldots, \gamma_{n+1}$. We want to estimate the measure of the set Δ_s with respect to the measure $\cos \eta \, d\eta$. Consider the intersection point $\mathbf{z}_{n+1}(\eta) = (z_{n+1}(\eta), u_{n+1}(\eta))$ of the segment $[\mathbf{x}(T^n \lambda), \mathbf{x}(T^{n+1} \lambda)]$ with ∂C_{jk}, which lies near the point $\mathbf{x}(T^{n+1} \lambda)$. Let

$$\Delta_s^{(n+1)} = \{ z_{n+1}(\eta) \mid \eta \in \Delta_s \}$$

Denote by $|\Delta_s^{(n+1)}|$ the diameter of the set $\Delta_s^{(n+1)}$. It is clear that

$$|\Delta_s^{(n+1)}| \leqslant h$$

where h is the period of the scatterer lattice along $\boldsymbol{\omega}_{jk}$. Consider similar sets $\Delta_s^{(m)}$ for all $m \leqslant n+1$. By point (ii) of Proposition 5.4 we have the recursive estimate

$$|\Delta_s^{(m)}| \leqslant C \frac{t_m + t_{m-1}}{(t_m)^3} |\Delta_s^{(m+1)}|$$

so

$$|\Delta_s^{(1)}| \leqslant h \prod_{m=1}^{n} \left[C \frac{t_m + t_{m-1}}{(t_m)^3} \right] \tag{5.19}$$

By (5.8), (5.10) we have that

$$\begin{aligned} \frac{dz_1}{\cos \eta \, d\eta} &= (d - y) \left[1 + \left(\frac{z_1 - x}{d - y} \right)^2 \right] \frac{1}{\sin(\alpha + \zeta_1)} \\ &= (d - y) \frac{(\tau_1)^2}{\sin(\alpha + \zeta_1)} \end{aligned}$$

where $d = d_{jk}$, $\mathbf{x}(\lambda) = (x, y)$, and $\tan \alpha = f_j'(x)$, where $y = f_j(x)$ is the equation of $\partial \Omega_j$ (we assume that Ω_j lies in the lower half-plane $\{ y \leqslant 0 \}$). In addition,

$$0 < \sin(\alpha + \zeta_1) = \sin \alpha \cos \zeta_1 + \cos \alpha \sin \zeta_1 \leqslant \sin \alpha + \frac{d - y}{\tau_1}$$

and $\tau_1 = |\mathbf{x}(T\lambda) - \mathbf{x}(\lambda)| = t_0 + o(1)$. Hence

$$\int_{\Delta_s} \cos\eta\, d\eta \leqslant \text{const} \cdot [\sin\alpha + (t_0)^{-1}](t_0)^{-2} \int_{\Delta_s^{(1)}} dz_1$$
$$= \text{const} \cdot [\sin\alpha + (t_0)^{-1}](t_0)^{-2}\, |\Delta_s^{(1)}|$$

Remark that the condition that the segment $[\mathbf{x}(\lambda), \mathbf{x}(T\lambda)]$ does not intersect the scatterer next to Ω_j implies that

$$|s| \leqslant \text{const} \cdot \sqrt{t_0}$$

so we get the inequality

$$\Pr\{V_{jk}^{+}(R; \gamma_1, \gamma_2,..., \gamma_{n+1})\}$$
$$= Z^{-1} \int ds \int_{\Delta_s} \cos\eta\, d\eta$$
$$\leqslant \text{const} \cdot \int_{\{|s| \leqslant \text{const}\,\sqrt{t_0}\}} ds[\sin\alpha + (t_0)^{-1}](t_0)^{-2}\, |\Delta_s^{(1)}|$$

Since $|\sin\alpha| \leqslant \text{const} \cdot |s|$, we come to the estimate

$$\Pr\{V_{jk}^{+}(R; \gamma_1, \gamma_2,..., \gamma_{n+1})\} \leqslant \text{const} \cdot (t_0)^{-3} \max_s |\Delta_s^{(1)}|$$

so in view of (5.19) we get that

$$\Pr\{V_{jk}^{+}(R; \gamma_1, \gamma_2,..., \gamma_{n+1})\} \leqslant \text{const} \cdot (t_0)^{-3} \prod_{m=1}^{n} \left[C \frac{t_m + t_{m-1}}{(t_m)^3} \right]$$

where

$$t_m = |\mathbf{x}_{\gamma_m} - \mathbf{x}_{\gamma_{m-1}}|$$

Due to (5.18), this implies that

$$\int_{V_{jk}^{+}(R; \gamma_1, \gamma_2,..., \gamma_{n+1}} |\mathbf{r}(\lambda)|\, |\mathbf{r}(T^n\lambda)|\, \mu_0(d\lambda)$$
$$\leqslant \text{const} \cdot (t_0)^{-2}\, t_n \prod_{m=1}^{n} \left[C \frac{t_m + t_{m-1}}{(t_m)_3} \right] \tag{5.20}$$

The condition that the segments $[\mathbf{x}(T^{m-1}\lambda), \mathbf{x}(T^m\lambda)]$, $[\mathbf{x}(T^m\lambda), \mathbf{x}(T^{m+1}\lambda)]$ do not intersect right and left of Ω_{γ_m} scatterers along the corridor C_{jk} implies that

$$C_1(t_m)^2 \geqslant t_{m+1} \geqslant C_2 \sqrt{t_m} \tag{5.21}$$

Let us enumerate all the scatterers Ω_γ touching the corridor C_{jk} in the following way. The enumeration is a one-to-one map i: $\{\Omega_\gamma\} \to \mathbb{Z}$ which is determined uniquely by the properties:

(i) $i(\Omega_\gamma) > i(\Omega_{\gamma'})$ if $x_\gamma > x_{\gamma'}$ or $x_\gamma = x_{\gamma'}$, $y_\gamma > y_{\gamma'}$, where $(x_\gamma, y_\gamma) = \mathbf{x}_\gamma = \partial\Omega_\gamma \cap \partial C_{jk}$ (monotonicity).

(ii) $i(\Omega_j) = 0$ (normalization).

One can see that $\exists C_3 > C_4 > 0$ such that

$$C_3 \, |i(\Omega_\gamma) - i(\Omega_{\gamma'})| \geqslant |\mathbf{x}_\gamma - \mathbf{x}_{\gamma'}| \geqslant C_4 \, |i(\Omega_\gamma) - i(\Omega_{\gamma'})| \tag{5.22}$$

According to this new enumeration, we have from (5.20), (5.22) that

$$\begin{aligned} &\int_{V_{jk}^+(R;\gamma_1,\gamma_2,\ldots,\gamma_{n+1})} |\mathbf{r}(\lambda)| \, |\mathbf{r}(T^n\lambda)| \, \mu_0(d\lambda) \\ &\qquad \leqslant \text{const} \cdot (i_0)^{-2} \, i_n \prod_{m=1}^{n} \left[C_0 \frac{i_m + i_{m-1}}{(i_m)^3} \right] \end{aligned} \tag{5.23}$$

where $i_m = i(\Omega_{\gamma_{m+1}}) - i(\Omega_{\gamma_m})$, $m = 0, \ldots, n$. Hence

$$\begin{aligned} &\sum_{\{\gamma_1,\ldots,\gamma_{n+1}\} \in G_n^{(0)}} \int_{V_{jk}^+(R;\gamma_1,\gamma_2,\ldots,\gamma_{n+1})} |\mathbf{r}(\lambda)| \, |\mathbf{r}(T^n\lambda)| \, \mu_0(d\lambda) \\ &\qquad \leqslant \text{const} \cdot \sum_{I_n} (i_0)^{-2} \, i_n \prod_{m=1}^{n} \left[C_0 \frac{i_m + i_{m-1}}{(i_m)^3} \right] \end{aligned}$$

where in view of (5.21), (5.22) the latter sum is taken over the set

$$\begin{aligned} I_n = \{ (i_0, i_1, \ldots, i_n) \mid i_0, i_n \geqslant R; \; C_5(i_{m-1})^2 \geqslant i_m \\ \geqslant C_6(i_{m-1})^{1/2}, m = 1, \ldots, n \} \end{aligned}$$

where $C_5 = C_1 C_3^2 / C_4$, $C_6 = C_2 \sqrt{C_4} / C_3$. Write

$$S_n = \sum_{I_n} (i_0)^{-2} \, i_n \prod_{m=1}^{n} \left[C_0 \frac{i_m + i_{m-1}}{(i_m)^3} \right]$$

Since

$$\sum_{i_n = [C_6(i_{n-1})^{1/2}]}^{[C_5 i_{n-1}^2]+1} C_0 \frac{i_n + i_{n-1}}{i_n^2} \leqslant C_0(C_6 + 1)(i_{n-1})^{1/2} \leqslant i_{n-1}$$

we get that $S_n \leqslant S_{n-1}$. So $S_n \leqslant S_1 < \infty$. Thus we have proven that

$$\sum_{\{\gamma_1,\ldots,\gamma_{n+1}\} \in G_n^{(0)}} \int_{V_{jk}^+(R;\gamma_1,\gamma_2,\ldots,\gamma_{n+1})} |\mathbf{r}(\lambda)| \, |\mathbf{r}(T^n\lambda)| \, \mu_0(d\lambda) < \infty$$

This means that we have estimated the contribution to $\int_{V_{jk}^+(R)}$ from the trajectories $\mathbf{x}(\lambda)$, $\mathbf{x}(T\lambda)$,..., $\mathbf{x}(T^{n+1}\lambda)$ which go at each step from one side of the corridor C_{jk} to another.

Consider now an arbitrary trajectory $\mathbf{x}(\lambda)$, $\mathbf{x}(T\lambda)$,..., $\mathbf{x}(T^{n+1}\lambda)$ which goes along the corridor C_{jk} so that the reflections shown in Fig. 3 are permitted. Let $\mathbf{x}(T^m\lambda)\in\partial\Omega_{\gamma_m}$, $m=0, 1,..., n+1$, and for some $m\geqslant 3$ the scatterers $\Omega_{\gamma_{m-3}}$, Ω_{γ_m} lie on the other side of the corridor C_{jk} than $\Omega_{\gamma_{m-2}}$, $\Omega_{\gamma_{m-1}}$ do. Recall that by point (iii) of Proposition 5.5 we have the following relation:

$$\frac{dz_m}{dz_{m-2}}=\frac{4\tau_{m-1}(\tau_m)^2}{\rho_{m-1}\rho_{m-2}\tau_{m-2}\cos\eta_{m-2}\cos\eta_{m-1}}\times[1+O((\tau_m)^{-1}+(\tau_{m-2})^{-1})]$$

where $\tau_m=|\mathbf{x}(T^m\lambda)-\mathbf{x}(T^{m-1}\lambda)|$ and η_m is the reflection angle at the point $\mathbf{x}(T^m\lambda)$. It is easy to see that

$$\eta_{m-2}+\eta_{m-1}+\frac{\zeta_{m-2}+\zeta_m}{2}=\pi$$

where ζ_m is the angle without sign between the vector $\mathbf{r}(T^{m-1}\lambda)=\mathbf{x}(T^m\lambda)-\mathbf{x}(T^{m-1}\lambda)$ and $\boldsymbol{\omega}_{jk}$. Hence

$$\sin\frac{\zeta_{m-2}+\zeta_m}{2}=\sin(\eta_{m-2}+\eta_{m-1})$$

so

$$\begin{aligned}\cos\eta_{m-2}\cos\eta_{m-1}&\leqslant\tfrac{1}{4}(\cos\eta_{m-2}+\cos\eta_{m-1})^2\\&=[(\tau_m)^{-1}+(\tau_{m-2})^{-1}]^2[1+O((\tau_m)^{-1}+(\tau_{m-2})^{-1})]\end{aligned}$$

Thus

$$\begin{aligned}\frac{dz_m}{dz_{m-2}}&\geqslant\frac{16\tau_{m-1}(\tau_m)^2}{\rho_{m-1}\rho_{m-2}\tau_{m-2}[(\tau_m)^{-1}+(\tau_{m-2})^{-1}]^2}\times[1+O((\tau_m)^{-1}+(\tau_{m-2})^{-1})]\\&\geqslant\text{const}\cdot\frac{(\tau_m)^3}{\tau_m+\tau_{m-2}}\cdot\frac{\tau_m\tau_{m-2}}{\tau_m+\tau_{m-2}}\end{aligned}$$

Comparing this inequality with point (ii) of Proposition 5.5, one can see that the expansion coefficient dz_m/dz_{m-2} of the double reflection shown in Fig. 3 is greater than the expansion coefficient dz_m/dz_{m-1} of a single reflection. This means that all the estimates we used for trajectories with only single reflections remain valid for general trajectories with both single and double reflections. So we get that

$$\int_{V_{jk}^{\pm}(R)} |\mathbf{r}(\lambda)|\ |\mathbf{r}(T^n\lambda)|\ \mu_0(d\lambda) < \infty$$

Theorem 5.6 is proved.

6. STATISTICAL BEHAVIOR OF TRAJECTORIES IN DISCRETE DYNAMICS

Equation (3.9) implies that

$$\begin{aligned}\langle |\mathbf{x}(T^n\lambda)-\mathbf{x}(\lambda)|^2\rangle &= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\langle(\mathbf{r}(T^i\lambda),\mathbf{r}(T^j\lambda))\rangle\\ &= n\langle|\mathbf{r}(\lambda)|^2\rangle + 2\sum_{j=1}^{n-1}(n-j)\langle(\mathbf{r}(\lambda),\mathbf{r}(T^j\lambda))\rangle \qquad (6.1)\end{aligned}$$

so

$$\begin{aligned}D &= \lim_{n\to\infty}\frac{\langle|\mathbf{x}(T^n\lambda)-\mathbf{x}(\lambda)|^2\rangle}{4n}\\ &= \frac{1}{4}\langle|\mathbf{r}(\lambda)|^2\rangle + \frac{1}{2}\sum_{n=1}^{\infty}\langle(\mathbf{r}(\lambda),\mathbf{r}(T^n\lambda))\rangle \qquad (6.2)\end{aligned}$$

which is a discrete variant of the Einstein–Green–Kubo relation (1.5). In the case under consideration

$$\langle|\mathbf{r}(\lambda)|^2\rangle = \infty \qquad (6.3)$$

(by Proposition 4.3) and

$$|\langle(\mathbf{r}(\lambda),\mathbf{r}(T^n\lambda))\rangle| < \infty \qquad (6.4)$$

for any $n \geqslant 1$ (by Theorem 5.6). A strong generalization of the last result is the following conjecture.

Conjecture I. There exist $C > 0$, $\kappa > 0$, and $1 > \gamma > 0$ such that for $n \geqslant 1$,

$$|\langle(\mathbf{r}(\lambda),\mathbf{r}(T^n\lambda))\rangle| \leqslant C\exp(-\kappa n^{\gamma}) \qquad (6.5)$$

For a periodic Lorentz gas with a finite horizon this was established in ref. 8. Numerical results reported in ref. 16 give for a square lattice of circular scatterers the asymptotics

$$\langle(\mathbf{v}(\lambda), \mathbf{v}(T^n\lambda))\rangle \sim (-1)^n \exp(-\kappa n^\gamma) \tag{6.6}$$

with $\gamma = 0.86 \pm 0.06$ (apparently independent of the radius of scatterers). Here $\mathbf{v}(T^n\lambda) = \mathbf{r}(T^n\lambda)/|\mathbf{r}(T^n\lambda)|$ is the velocity of the particle between the nth and $(n+1)$th collisions. [It is noteworthy that the fast decrease of the velocity autocorrelation function in (6.6) implies that the series $\sum_{n=0}^{\infty} |\langle(\mathbf{v}(\lambda), \mathbf{v}(T^n(\lambda)))\rangle|$ is convergent. However, the convergence of this series *does not imply* that the discrete diffusion coefficient D in (6.2) is finite (see in this connection refs. 9, 16), because the equality $D = \infty$ is related to $\langle|\mathbf{r}(\lambda)|^2\rangle = \infty$ and not to the divergence of the series of correlations.]

In view of the relations (6.3), (6.4) we have from (6.1) that $\langle|\mathbf{x}(T^n\lambda) - \mathbf{x}(\lambda)|^2\rangle = \infty$. To find the right normalization of

$$\mathbf{x}(T^n\lambda) - \mathbf{x}(\lambda) = \sum_{j=0}^{n-1} \mathbf{r}(T^j\lambda)$$

for $n \to \infty$ consider the sum

$$\mathbf{S}_n = \sum_{j=0}^{n-1} \xi_j$$

of independent identically distributed random variables ξ_j with

$$\xi_j \stackrel{d}{=} \mathbf{r}(T^j\lambda) \stackrel{d}{=} \mathbf{r}(\lambda)$$

The question we are interested in is: What is a right normalization for $\mathbf{S}_n$ and what is the limit distribution of normalized $\mathbf{S}_n$? Since the variance of $\mathbf{r}(\lambda)$ is infinite, it is not the classical case with the normalization in square root of the number of random variables, but still the variance diverges only logarithmically, so one may expect the appearance of some logarithmic corrections to the square root normalization.

To study this question in more detail, let us consider the characteristic function of $\mathbf{S}_n$,

$$\chi_n(\mathbf{t}) = \langle\exp[i(\mathbf{S}_n, \mathbf{t})]\rangle = [\chi(\mathbf{t})]^n$$

By Theorem 4.4

$$[\chi(\mathbf{t})]^n = \exp\left\{n\left[\sum_{j=1}^{N}\sum_{k=1}^{N_j} \alpha_{jk}(\mathbf{t}, \boldsymbol{\omega}_{jk})^2 \ln|(\mathbf{t}, \boldsymbol{\omega}_{jk})| + (A\mathbf{t}, \mathbf{t}) + O(|\mathbf{t}|^{9/4})\right]\right\}$$

so

$$\left\langle \exp\left[i\left(\frac{\mathbf{S}_n}{(n \ln n)^{1/2}}, \mathbf{t} \right)\right]\right\rangle$$

$$= \left[\chi\left(\frac{\mathbf{t}}{(n \ln n)^{1/2}} \right)\right]^n$$

$$= \exp\left\{ n \left[\sum_{j=1}^{N} \sum_{k=1}^{N_j} \frac{\alpha_{jk}(\mathbf{t}, \boldsymbol{\omega}_{jk})^2}{n \ln n} \ln \left| \left(\frac{\mathbf{t}}{(n \ln n)^{1/2}}, \boldsymbol{\omega}_{jk} \right) \right| \right.\right.$$

$$\left.\left. + \frac{(A\mathbf{t}, t)}{n \ln n} + \frac{O(|\mathbf{t}|^{9/4})}{(n \ln n)^{9/8}} \right]\right\}$$

$$= \exp\left[-\frac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N_j} \alpha_{jk}(\mathbf{t}, \boldsymbol{\omega}_{jk})^2 + O\left(\frac{\ln \ln n}{\ln n} \right)\right]$$

so

$$\lim_{n \to \infty} \left\langle \exp\left[i\left(\frac{\mathbf{S}_n}{(n \ln n)^{1/2}}, \mathbf{t} \right)\right]\right\rangle = \exp\left\{ -\frac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N_j} \alpha_{jk}(\mathbf{t}, \boldsymbol{\omega}_{jk})^2 \right\}$$

Thus, the characteristic function of

$$\frac{\mathbf{S}_n}{(n \ln n)^{1/2}} = \frac{\sum_{j=0}^{n-1} \boldsymbol{\xi}_j}{(n \ln n)^{1/2}}$$

converges to the Gaussian function

$$\exp\left\{ -\frac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N_j} \alpha_{jk}(\mathbf{t}, \boldsymbol{\omega}^{jk})^2 \right\}$$

so

$$\lim_{n \to \infty} \frac{\sum_{j=0}^{n-1} \boldsymbol{\xi}_j}{(n \ln n)^{1/2}} = \boldsymbol{\xi} \tag{6.7}$$

where $\boldsymbol{\xi}$ is a Gaussian random variable with zero mean and the covariance matrix

$$\langle \xi_l \xi_m \rangle = \sum_{j=1}^{N} \sum_{k=1}^{N_j} \alpha_{jk} \omega_{jkl} \omega_{jkm}, \qquad l, m = 1, 2$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2)$, $\boldsymbol{\omega}_{jk} = (\omega_{jk1}, \omega_{jk2})$. In (6.7) and later we consider the convergence of random variables in weak topology.

Now we formulate our main result. Its proof is based on Conjectures I′, II, and III, which are formulated below and concern some properties of the dependence of the vectors $\mathbf{r}(T^j\lambda)$.

Theorem 6.1. Let the distribution of $\lambda \in \Lambda_0$ be the Liouville measure $\mu_0(d\varphi\, dr) = Z^{-1} \cos\varphi\, d\varphi\, dr$. Assume that Conjectures I′, II, and III which are formulated below are true. Then

$$\lim_{n\to\infty} \frac{\mathbf{x}(T^n\lambda) - \mathbf{x}(\lambda)}{(n \ln n)^{1/2}} = \boldsymbol{\xi} \tag{6.8}$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2)$ is a Gaussian random variable with zero mean and the covariance matrix

$$D_{lm} = \langle \xi_l \xi_m \rangle = \sum_{j=1}^{N} \sum_{k=1}^{N_j} \alpha_{jk} \omega_{jkl} \omega_{jkm}, \qquad l, m = 1, 2 \tag{6.9}$$

where

$$\alpha_{jk} = \frac{1}{4 \sum_{l=1}^{N} |\partial \Omega_l|} h_{jk} d_{jk}^2, \qquad h_{jk} = h_{jk}^+ + h_{jk}^-$$

and $\boldsymbol{\omega}_{jk} = (\omega_{jk1}, \omega_{jk2})$ is the direction of the corridor C_{jk}; d_{jk} is the width of this corridor and $h_{jk}^{\pm}$ are the distances from the scatterer Ω_j to its neighbors along the corridor C_{jk}.

Remark. In a stronger version of the theorem one may think of any distribution $\mu(d\lambda)$ of λ on Λ_0 which is absolutely continuous with respect to $\mu_0(d\lambda)$ with

$$0 < C_0 < \frac{\mu(d\lambda)}{\mu_0(d\lambda)} < C_1$$

A statement is that the limit (6.8) exists for any measure $\mu(d\lambda)$ satisfying these properties [and so it does not depend on $\mu(d\lambda)$], but we will not discuss this stronger form of the theorem here.

Proof. Actually we state that the asymptotic behavior for $n \to \infty$ of the sum $\sum_{j=0}^{n-1} \mathbf{r}(T^j\lambda)$ is as if $\mathbf{r}(T^j\lambda)$were independent random variables. Some formal explanation of that comes from the fact that the correlation coefficient of $\mathbf{r}(T^i\lambda)$ and $\mathbf{r}(T^j\lambda)$ is

$$K(i, j) \equiv \frac{\langle (\mathbf{r}(T^i\lambda), \mathbf{r}(T^j\lambda)) \rangle}{\langle |\mathbf{r}(T^i\lambda)|^2 \rangle \langle |\mathbf{r}(T^j\lambda)|^2 \rangle^{1/2}} = 0, \qquad i \neq j$$

because the numerator is finite while the denominator is infinite in the last ratio. In the rigorous proof, however, we need much more difficult and fine estimates.

Introduce a cutoff for $\mathbf{r}(\lambda)$:

$$\mathbf{r}^{(n)}(\lambda)=\begin{cases}\mathbf{r}(\lambda), & \text{if } \quad |\mathbf{r}(\lambda)|\leqslant (n\ln\ln n)^{1/2}\\ 0 & \text{otherwise}\end{cases}$$

Since

$$\Pr\{|\mathbf{r}(\lambda)|\geqslant (n\ln\ln n)^{1/2}\}\sim \text{const}\cdot(n\ln\ln n)^{-1} \tag{6.10}$$

we have that

$$\Pr\left\{\sum_{j=0}^{n-1}\mathbf{r}(T^j\lambda)=\sum_{j=0}^{n-1}\mathbf{r}^{(n)}(T^j\lambda)\right\}\geqslant 1-n\cdot\frac{\text{const}}{n\ln\ln n}$$
$$=1-\frac{\text{const}}{\ln\ln n}$$

so

$$\lim_{n\to\infty}\Pr\left\{\sum_{j=0}^{n-1}\mathbf{r}(T^j\lambda)=\sum_{j=0}^{n-1}\mathbf{r}^{(n)}(T^j\lambda)\right\}=1$$

Hence to establish (6.8) it is enough to show that

$$\lim_{n\to\infty}\frac{\sum_{j=0}^{n-1}\mathbf{r}^{(n)}(T^j\lambda)}{(n\ln n)^{1/2}}=\xi \tag{6.11}$$

Let us calculate the asymptotics of the covariance matrix of $\mathbf{r}^{(n)}(\lambda)=(r_1^{(n)}(\lambda), r_2^{(n)}(\lambda))$ when $n\to\infty$. We have

$$\langle r_l^{(n)}(\lambda)\, r_m^{(n)}(\lambda)\rangle=\int_{|\mathbf{r}|\leqslant(n\ln\ln n)^{1/2}} r_l r_m \nu^0(d\mathbf{r}), \qquad l, m=1, 2$$

where $\nu^0(d\mathbf{r})$ is the distribution of $\mathbf{r}(\lambda)$. So by Proposition 4.2

$$\langle r_l^{(n)}(\lambda)\, r_m^{(n)}(\lambda)\rangle$$
$$=\sum_{j=1}^{N}\sum_{k=1}^{N_j}\sum_{\pm}\int_1^{(n\ln\ln n)^{1/2}} Z^{-1}h_{jk}^{\pm}(d_{jk})^2\frac{2}{R^3}(R\omega_{jkl})(R\omega_{jkm})\,dR+O(1)$$
$$=\left(\sum_{j=1}^{N}\sum_{k=1}^{N_j}\alpha_{jk}\omega_{jkl}\omega_{jkm}\right)\ln n\left[1+O\left(\frac{\ln\ln n}{\ln n}\right)\right]$$
$$=D_{lm}\ln n\left[1+O\left(\frac{\ln\ln n}{\ln n}\right)\right]$$

so that

$$\lim_{n\to\infty} \frac{\langle r_l^{(n)}(\lambda)\, r_m^{(n)}(\lambda)\rangle}{\ln n} = D_{lm} = \langle \xi_l \xi_m \rangle, \quad l, m = 1, 2 \tag{6.12}$$

This shows that the covariance matrix of $\mathbf{r}^{(n)}(\lambda)/(\ln n)^{1/2}$ converges for $n\to\infty$ to that of ξ.

Let us estimate the covariance matrix of

$$\sum_{j=0}^{n-1} \mathbf{r}^{(n)}(T^j\lambda) \equiv \mathbf{S}^{(n)}(\lambda) = (S_1^{(n)}(\lambda), S_2^{(n)}(\lambda))$$

We have

$$\begin{aligned}\langle S_l^{(n)}(\lambda)\, S_m^{(n)}(\lambda)\rangle &= \sum_{i=0}^{n-1}\sum_{j=1}^{n-1} \langle r_l^{(n)}(T^i\lambda)\, r_m^{(n)}(T^j\lambda)\rangle \\ &= \sum_{j=-n+1}^{n-1} (n-|j|)\langle r_l^{(n)}(\lambda)\, r_m^{(n)}(T^j\lambda)\rangle \end{aligned} \tag{6.13}$$

A natural extension of Conjecture I is the following.

Conjecture I′. There exist $C>0$, $\kappa>0$, and $1>\gamma>0$ such that for $j\geqslant 1$ and $n\geqslant 1$,

$$|\langle r_l^{(n)}(\lambda)\, r_m^{(n)}(T^j\lambda)\rangle| \leqslant C\exp(-\kappa j^\gamma), \qquad l, m = 1, 2 \tag{6.14}$$

This implies that

$$\left|\sum_{j=1}^{n-1} (n-j)\langle r_l^{(n)}(\lambda)\, r_m^{(n)}(T^j\lambda)\rangle\right| \leqslant n \sum_{j=1}^{\infty} |\langle r_l^{(n)}(\lambda)\, r_m^{(n)}(T^j\lambda)\rangle| \leqslant C_0 n$$

Similarly,

$$\left|\sum_{j=-n+1}^{-1} (n-|j|)\langle r_l^{(n)}(\lambda)\, r_m^{(n)}(T^j\lambda)\rangle\right| \leqslant C_0 n$$

On the other hand, by (6.12),

$$n\langle r_l^{(n)}(\lambda)\, r_m^{(n)}(\lambda)\rangle = D_{lm} n \ln n \left[1 + O\left(\frac{\ln\ln n}{\ln n}\right)\right]$$

Hence

$$\frac{\langle \sum_{j=0}^{n-1} r_l^{(n)}(T^j\lambda) \sum_{j=0}^{n-1} r_m^{(n)}(T^j\lambda)\rangle}{n\ln n} = D_{lm}\left[1 + O\left(\frac{\ln\ln n}{\ln n}\right)\right] \tag{6.15}$$

so that the covariance matrix of $(n \ln n)^{-1/2} \sum_{j=0}^{n-1} \mathbf{r}^{(n)}(T^j\lambda)$ converges for $n \to \infty$ to that of ξ.

Now we use the celebrated Bernstein method of proving the central limit theorem for random processes. Here we follow ref. 8 with some natural modifications. Let us decompose the interval $\Delta = [0, n-1]$ into nonoverlapping subintervals

$$\Delta = \Delta_1^{(1)} \cup \Delta_1^{(2)} \cup \Delta_2^{(1)} \cup \Delta_2^{(2)} \cup \cdots \cup \Delta_{p-1}^{(2)} \cup \Delta_p^{(1)}$$

in such a way that the length $|\Delta_i^{(1)}|$ of each $\Delta_i^{(1)}$ except the last one is equal to $[n/\ln \ln n]$, while the length of each $\Delta_i^{(2)}$ is equal to $[n^\gamma]$, where

$$1/3 > \gamma > 0 \tag{6.16}$$

and $|\Delta_p^{(1)}| \leqslant [n/\ln \ln n]$. From the definition it follows that

$$(1/2) \ln \ln n < p < 2 \ln \ln n$$

We can write

$$\sum_{j=0}^{n-1} \mathbf{r}^{(n)}(T^j\lambda) = \sum_{s=1}^{p} \sum_{j \in \Delta_s^{(1)}} \mathbf{r}^{(n)}(T^j\lambda) + \sum_{s=1}^{p-1} \sum_{j \in \Delta_s^{(2)}} \mathbf{r}^{(n)}(T^j\lambda) \tag{6.17}$$

Let $m = (p-1)[n^\gamma]$. By (6.10)

$$\Pr\left\{\sum_{s=1}^{p-1} \sum_{j \in \Delta_s^{(2)}} \mathbf{r}^{(n)}(T^j\lambda) = \sum_{s=1}^{p-1} \sum_{j \in \Delta_s^{(2)}} \mathbf{r}^{(m)}(T^j\lambda)\right\} \geqslant 1 - \frac{\text{const}}{\ln \ln m}$$

so

$$\lim_{n \to \infty} \Pr\left\{\sum_{s=1}^{p-1} \sum_{j \in \Delta_s^{(2)}} \mathbf{r}^{(n)}(T^j\lambda) = \sum_{s=1}^{p-1} \sum_{j \in \Delta_s^{(2)}} \mathbf{r}^{(m)}(T^j\lambda)\right\} = 1 \tag{6.18}$$

In addition,

$$(n \ln n)^{-1/2} \sum_{s=1}^{p-1} \sum_{j \in \Delta_s^{(2)}} |\mathbf{r}^{(m)}(T^j\lambda)| \leqslant \frac{m(m \ln \ln m)^{1/2}}{(n \ln n)^{1/2}} \leqslant n^{(3/2)\gamma - 1/2}$$

By (6.16), $(3/2)\gamma - 1/2 < 0$, so

$$\lim_{n \to \infty} (n \ln n)^{-1/2} \sum_{s=1}^{p-1} \sum_{j \in \Delta_s^{(2)}} |\mathbf{r}^{(m)}(T^j\lambda)| = 0 \tag{6.19}$$

Relations (6.18), (6.19) show that the limit of $(n \ln n)^{-1/2} \sum_{j=0}^{n-1} \mathbf{r}^{(n)}(T^j\lambda)$ as $n \to \infty$ is the same as the limit of

$$\boldsymbol{\sigma}_n(\lambda) = \sum_{s=1}^{p} \boldsymbol{\sigma}_{ns}(\lambda)$$

where

$$\boldsymbol{\sigma}_{ns}(\lambda) = (n \ln n)^{-1/2} \sum_{j \in \Delta_s^{(1)}} \mathbf{r}^{(n)}(T^j \lambda) \tag{6.20}$$

Let us remark that different $\boldsymbol{\sigma}_{ns}(\lambda)$ are related to different parts of the trajectory $\{T^j\lambda, j = 0, 1, ..., n-1\}$ separated by the "corridors" $\Delta_s^{(2)}$. Because of strong ergodic properties of the map T, one may expect that $\boldsymbol{\sigma}_{ns}(\lambda)$ are almost independent for different s. For a periodic Lorentz gas with a finite horizon this statement was established in ref. 8. We formulate it in the form of a conjecture.

Conjecture II. For any fixed $\mathbf{t} \in \mathbb{R}^2$,

$$\lim_{n \to \infty} \left| \left\langle \exp\left\{ i\left(\mathbf{t}, \sum_{s=1}^{p} \boldsymbol{\sigma}_{ns}\right)\right\} \right\rangle - \prod_{s=1}^{p} \langle \exp\{ i(\mathbf{t}, \boldsymbol{\sigma}_{ns})\} \rangle \right| = 0 \tag{6.21}$$

Remark that $\prod_{s=1}^{p} \langle \exp\{i(\mathbf{t}, \boldsymbol{\sigma}_{ns})\}\rangle$ is the characteristic function of the sum $\xi_n = \sum_{s=1}^{p} \xi_{ns}$ of independent random variables ξ_{ns} with

$$\xi_{ns} \stackrel{d}{=} \boldsymbol{\sigma}_{ns}$$

The relation (6.21) implies that

$$\lim_{n \to \infty} \boldsymbol{\sigma}_n = \lim_{n \to \infty} \xi_n$$

if the latter limit does exist.

To show the existence of

$$\lim_{n \to \infty} \xi_n = \lim_{n \to \infty} \sum_{s=1}^{p} \xi_{ns} = \xi$$

we have to verify Lindeberg's condition (see, e.g., ref. 25): For any $t > 0$,

$$\lim_{s=1}^{p} \langle |\xi_{ns}^{(t)}|^2 \rangle = 0$$

where

$$\xi_{ns}^{(t)} = \begin{cases} 0 & \text{if} \quad |\xi_{ns}| \leqslant t \\ \xi_{ns} & \text{if} \quad |\xi_{ns}| > t \end{cases}$$

By Tchebyshev's inequality,

$$\langle |\xi_{ns}^{(t)}|^2 \rangle \leqslant \frac{\langle |\xi_{ns}|^4 \rangle}{t^2}$$

so our aim is to show that

$$\lim_{n\to\infty} \sum_{s=1}^{p} \langle |\xi_{ns}|^4 \rangle = 0 \tag{6.22}$$

We have

$$\begin{aligned}\langle |\xi_{ns}|^4 \rangle &= \langle |\boldsymbol{\sigma}_{ns}|^4 \rangle \\ &= (n \ln n)^{-2} \left\langle \left| \sum_{j\in\Delta_s^{(1)}} \mathbf{r}^{(n)}(T^j\lambda) \right|^4 \right\rangle \\ &= (n \ln n)^{-2} \sum_{j_1, j_2, j_3, j_4 \in \Delta_s^{(1)}} \langle (\mathbf{r}^{(n)}(T^{j_1}\lambda), \mathbf{r}^{(n)}(T^{j_2}\lambda)) \\ &\quad \times (\mathbf{r}^{(n)}(T^{j_3}\lambda), \mathbf{r}^{(n)}(T^{j_4}\lambda)) \rangle \end{aligned} \tag{6.23}$$

Let us first estimate in the last sum the sum of diagonal terms. Since

$$\langle |\mathbf{r}^{(n)}(\lambda)|^4 \rangle = \int_{|\mathbf{r}| \leqslant (n \ln \ln n)^{1/2}} |\mathbf{r}|^4 \, v^0(d\mathbf{r}) \leqslant C \int_0^{(n \ln \ln n)^{1/2}} r^4 \frac{dr}{r^3} \leqslant Cn \ln \ln n \tag{6.24}$$

we have that

$$\sum_{j\in\Delta_s^{(1)}} \langle |\mathbf{r}^{(n)}(Tj\lambda)|^4 \rangle = |\Delta_s^{(1)}| \, \langle |\mathbf{r}^{(n)}(\lambda)|^4 \rangle \geqslant C \frac{n}{\ln \ln n} n \ln \ln n = Cn^2 \tag{6.25}$$

To estimate off-diagonal terms, we formulate the following general conjecture. Let

$$\operatorname{diam}\{j_1,..., j_k\} = \max_{1 \leqslant l, m \leqslant k} |j_l - j_m|$$

Conjecture III (Estimates of truncated correlation functions of the fourth order). There exist $C>0$, $\kappa>0$, and $1>\gamma>0$ such that for $n \geqslant 1$, $l_1, l_2, l_3, l_4 = 1, 2$, and pairwise different $j_1, j_2, j_3, j_4 \in \mathbb{Z}$,

$$\begin{aligned} &|\langle r_{l_1}^{(n)}(T^{j_1}\lambda)\, r_{l_2}^{(n)}(T^{j_2}\lambda)\, r_{l_3}^{(n)}(T^{j_3}\lambda)\, r_{l_4}^{(n)}(T^{j_4}\lambda)\rangle| \\ &\quad \leqslant C_n \exp[-\kappa(\operatorname{diam}\{j_1, j_2, j_3, j_4\})^\gamma] \\ &\quad |\langle r_{l_1}^{(n)}(T^{j_1}\lambda)\, r_{l_2}^{(n)}(T^{j_2}\lambda)[r_{l_3}^{(n)}(T^{j_3}\lambda)]^2\rangle \\ &\quad - \langle r_{l_1}^{(n)}(T^{j_1}\lambda)\, r_{l_2}^{(n)}(T^{j_2}\lambda)\rangle \langle r_{l_3}^{(n)}(T^{j_3}\lambda)\rangle^2| \\ &\quad \leqslant C_n \exp[-\kappa(\operatorname{diam}\{j_1, j_2, j_3\})^\gamma] \\ &|\langle [r_{l_1}^{(n)}(T^{j_1}\lambda)]^2\, [r_{l_2}^{(n)}(T^{j_2}\lambda)]^2\rangle - \langle r_{l_1}^{(n)}(T^{j_1}\lambda)\rangle^2 \, \langle r_{l_2}^{(n)}(T^{j_2}\lambda)\rangle^2| \\ &\quad \leqslant C_n \exp(-\kappa\, |j_1 - j_2|^\gamma) \\ &\quad |\langle [r_{l_1}^{(n)}(T^{j_1}\lambda)]^3\, [r_{l_2}^{(n)}(T^{j_2}\lambda)]\rangle| \leqslant C_n \exp(-\kappa\, |j_1 - j_2|^\gamma) \end{aligned}$$

where $C_n = C\langle |\mathbf{r}^{(n)}(\lambda)|^4 \rangle$.

This conjecture is closely related to the thermodynamic formalism developed for billiard systems in refs. 7, 8, and 24. The thermodynamic formalism enables us to represent the invariant Liouville measure as a Gibbs measure on $\mathbb{Z}$ with the action of T as a shift of $\mathbb{Z}$ in 1. In this context the estimates formulated in Conjecture III are just the usual estimates of truncated correlation functions of a one-dimensional Gibbs measure.

The number of quadruples $\{j_1, j_2, j_3, j_4\}$ with $\operatorname{diam}\{j_1, j_2, j_3, j_4\} \leqslant j$ and with fixed j_1 does not exceed Cj^3, so we get from Conjecture III and (6.24) that

$$\begin{aligned}\langle |\xi_{ns}|^4 \rangle &= (n \ln n)^{-2} \sum_{j_1, j_2, j_3, j_4 \in \Delta_s^{(1)}} \langle [\mathbf{r}^{(n)}(T^{j_1}\lambda), \mathbf{r}^{(n)}(T^{j_2}\lambda)] \\ &\quad \times [\mathbf{r}^{(n)}(T^{j_3}\lambda), \mathbf{r}^{(n)}(T^{j_4}\lambda)] \rangle \\ &\leqslant (n \ln n)^{-2} C_0 n \ln \ln n \sum_{j_1 \in \Delta_s^{(1)}} \sum_{j=1}^{\infty} j^3 \exp(-\kappa j^\gamma) \leqslant C_1 (\ln n)^{-2}\end{aligned}$$

So

$$\sum_{s=1}^{p} \langle |\xi_{ns}|^4 \rangle \leqslant C_1 \ln \ln n \, (\ln n)^{-2}$$

which implies (6.22).

This finishes the proof of Theorem 6.1.

7. STATISTICAL BEHAVIOR OF TRAJECTORIES IN CONTINUOUS-TIME DYNAMICS

The aim of this section is to extend the main formula (6.8) to the continuous-time dynamics. In what follows, to distinguish $\mathbf{x}(t)$ from $\mathbf{x}(\lambda)$ we shall denote it $\mathbf{x}_t$. As before, we denote by $\mathbf{x}(\lambda) = \mathbf{x}(j, s, \eta)$ the point at $\partial \Omega_j$ with natural coordinate s. Let $\mathbf{v}(\lambda) = \mathbf{v}(j, s, \eta)$ be the velocity vector of the particle between $\mathbf{x}(\lambda)$ and $\mathbf{x}(T\lambda)$. The angle between $\mathbf{v}(\lambda)$ and the normal vector $\mathbf{n}$ at $\mathbf{x}(\lambda)$ is equal to η, and $|\mathbf{v}(\lambda)| = 1$.

We shall assume that the initial conditions $\mathbf{x}_t|_{t=0}$ and $\mathbf{v}_t|_{t=0}$ are random and

$$\mathbf{x}_t|_{t=0} = \mathbf{x}(\lambda), \qquad \mathbf{v}_t|_{t=0} = \mathbf{v}(\lambda) \tag{7.1}$$

where λ is a random point in Λ_0 distributed according to the Liouville measure $\mu_0(d\lambda) = Z^{-1} \cos \varphi \, d\varphi \, ds$. Our main result in the present section concerns the description of the statistical behavior of $\mathbf{x}_t$ as $t \to \infty$. We shall show that Theorem 6.1 implies that

$$\lim_{t \to \infty} \frac{\mathbf{x}_t - \mathbf{x}_0}{(t \ln t)^{1/2}} = \frac{\xi}{\sqrt{\tau}} \tag{7.2}$$

where ξ is the same as in Theorem 6.1 and

$$\tau = \langle |\mathbf{r}(\lambda)| \rangle = \int_{\Lambda_0} |\mathbf{r}| \, \nu^0(d\mathbf{r}) \tag{7.3}$$

is the spatial mean of the length of free motion. Let us remark that by Proposition 4.3, τ is finite.

We shall use the following abstract lemma.

Lemma 7.1. Assume that for any $t \geqslant 0$, $\mathbf{a}_t \in \mathbb{R}^k$ and $b_t \in \mathbb{R}$ are random variables on a probability space (X, B, μ) and the probability distribution of $\mathbf{a}_t$ converges as $t \to \infty$ in weak sense to a probability distribution $\nu(d\mathbf{a})$, while b_t converges as $t \to \infty$ almost surely to a constant b^0. Then the probability distribution of $b_t \mathbf{a}_t$ converges as $t \to \infty$ to that of $b^0 \mathbf{a}$, where the distribution of $\mathbf{a}$ is $\nu(d\mathbf{a})$.

Proof. We have to show[26] that for any $\varphi \in C^1(\mathbb{R}^k)$,

$$\lim_{t \to \infty} \int \varphi(b_t(x)\, \mathbf{a}_t(x))\, \mu(dx) = \int \varphi(b^0 \mathbf{a})\, \nu(d\mathbf{a}) \tag{7.4}$$

Denote

$$X_\varepsilon^t = \{x \in X \mid |b_t(x) - b^0| < \varepsilon,\ |\mathbf{a}_t(x)| < \varepsilon^{-1/2}\}$$

Then for any $\varepsilon > 0$ there exists $t_0 > 0$ such that $\mu(X_\varepsilon^t) > 1 - \varepsilon$ when $t > t_0$. For $x \in X_\varepsilon^t$,

$$|\varphi(b_t(x)\, \mathbf{a}_t(x)) - \varphi(b^0 \mathbf{a}_t(x))| \leqslant \|\varphi\|_{C^1(\mathbb{R}^k)}\, \varepsilon^{1/2}$$

so

$$\lim_{t \to \infty} \left| \int \varphi(b_t(x)\, \mathbf{a}_t(x))\, \mu(dx) - \int \varphi(b^0 \mathbf{a}_t(x))\, \mu(dx) \right| = 0 \tag{7.5}$$

On the other hand, the weak convergence of $\mathbf{a}_t(x)$ implies that

$$\lim_{t \to \infty} \int \varphi(b^0 \mathbf{a}_t(x))\, \mu(dx) = \int \varphi(b^0 \mathbf{a})\, \nu(d\mathbf{a}) \tag{7.6}$$

Since (7.5), (7.6) imply (7.4), Lemma 7.1 is proved.

Let us denote by $t_n = t_n(\lambda)$ the time of the nth reflection at the trajectory $\mathbf{x}_t$ with the initial conditions (7.1), so that

$$\mathbf{x}_{t_n} = \mathbf{x}(T^n \lambda)$$

Since between collisions the particle moves freely with velocity 1,

$$t_n - t_{n-1} = |\mathbf{x}(T^n\lambda) - \mathbf{x}(T^{n-1}\lambda)| = |\mathbf{r}(T^{n-1}\lambda)|$$

hence

$$t_n = \sum_{j=0}^{n-1} |\mathbf{r}(T^j\lambda)|$$

By the ergodic theorem we get that almost surely

$$\lim_{n\to\infty} \frac{t_n}{n} = \lim_{n\to\infty} n^{-1} \sum_{j=0}^{n-1} |\mathbf{r}(T^j\lambda)| = \langle |\mathbf{r}(\lambda)| \rangle \equiv \tau \tag{7.7}$$

This implies that almost surely

$$\lim_{n\to\infty} \frac{t_n \ln t_n}{n \ln n} = \tau$$

and

$$\lim_{n\to\infty} \frac{t_{n+1} \ln t_{n+1}}{t_n \ln t_n} = 1$$

Assume that $t_n \leqslant t < t_{n+1}$. Then by Lemma 7.1 and the last two equations,

$$\lim_{n\to\infty} \frac{\mathbf{x}_t - \mathbf{x}_0}{(t \ln t)^{1/2}} = \lim_{n\to\infty} \frac{\mathbf{x}_t - \mathbf{x}_0}{(t_n \ln t_n)^{1/2}} = \tau^{-1/2} \lim_{n\to\infty} \frac{\mathbf{x}_t - \mathbf{x}_0}{(n \ln n)^{1/2}} \tag{7.8}$$

Let us remark that if $t_n \leqslant t < t_{n+1}$, then $\mathbf{x}_t$ lies in the segment $[\mathbf{x}(T^n\lambda), \mathbf{x}(T^{n+1}\lambda)]$, so $|\mathbf{x}_t - \mathbf{x}(T^n\lambda)| \leqslant |\mathbf{r}(T^n\lambda)|$. Therefore almost surely

$$\lim_{n\to\infty} \frac{\mathbf{x}_t - \mathbf{x}(T^n\lambda)}{(n \ln n)^{1/2}} = 0$$

so

$$\lim_{n\to\infty} \frac{\mathbf{x}_t - \mathbf{x}_0}{(n \ln n)^{1/2}} = \lim_{n\to\infty} \frac{\mathbf{x}(T^n\lambda) - \mathbf{x}(\lambda)}{(n \ln n)^{1/2}} = \xi$$

so by (7.8)

$$\lim_{t\to\infty} \frac{\mathbf{x}_t - \mathbf{x}_0}{(t \ln t)^{1/2}} = \frac{\xi}{\sqrt{\tau}}$$

Formula (7.2) is proved.

8. DISCUSSION

We have shown in Theorem 6.1 that in a periodic Lorentz gas with an infinite horizon

$$\lim_{n\to\infty} \frac{\mathbf{x}_n - \mathbf{x}_0}{(n \ln n)^{1/2}} = \xi \tag{8.1}$$

(discrete dynamics) and

$$\lim_{t\to\infty} \frac{\mathbf{x}(t) - \mathbf{x}(0)}{(t \ln t)^{1/2}} = \frac{\xi}{\sqrt{\tau}} \tag{8.2}$$

(continuous-time dynamics) where $\xi = (\xi_1, \xi_2)$ is a Gaussian random variable with zero mean and the covariance matrix

$$D_{lm} = \langle \xi_l \xi_m \rangle = \sum_{j=1}^{N} \sum_{k=1}^{N_j} \alpha_{jk} \omega_{jkl} \omega_{jkm}, \qquad l, m = 1, 2 \tag{8.3}$$

where

$$\alpha_{jk} = \frac{1}{4 \sum_{l=1}^{N} |\partial \Omega_l|} h_{jk} d_{jk}^2, \qquad h_{jk} = h_{jk}^+ + h_{jk}^- \tag{8.4}$$

and $\boldsymbol{\omega}_{jk} = (\omega_{jk1}, \omega_{jk2})$ is the direction of the corridor C_{jk}; d_{jk} is the width of this corridor and $h_{jk}^{\pm}$ are the distances from the scatterer Ω_j to its neighbors along the corridor C_{jk}, and

$$\tau = \langle |\mathbf{r}(\lambda)| \rangle = \int_{\Lambda_0} |\mathbf{r}|\, \nu^0(d\mathbf{r})$$

is the mean time of free motion. Our proof of Theorem 6.1 was based on Conjectures I′, II, III, which are formulated in Section 6. They concern some estimates of the dependence of the free motion vectors $\mathbf{r}(T^j\lambda)$, $j \in \mathbb{Z}$.

Let us define the quadratic form generated by the covariance matrix,

$$A(\mathbf{z}, \mathbf{z}) = \sum_{l,m=1,2} D_{lm} z_l z_m$$

It follows from (8.3) that

$$A(\mathbf{z}, \mathbf{z}) = \sum_{j=1}^{N} \sum_{k=1}^{N_j} \alpha_{jk} (\boldsymbol{\omega}, \mathbf{z})^2$$

so that the covariance matrix is positive if there exist two nonparallel corridors.

Equation (8.2) shows that the diffusion coefficient

$$D = \lim_{t \to \infty} \frac{1}{4t} \langle |\mathbf{x}(t) - \mathbf{x}(0)|^2 \rangle$$

is infinite and so we have a sort of "super"-diffusive behavior of trajectories $\mathbf{x}(t)$ as $t \to \infty$. In this context the number

$$D = \frac{1}{4} \left\langle \left| \lim_{t \to \infty} \frac{\mathbf{x}(t) - \mathbf{x}(0)}{(t \ln t)^{1/2}} \right|^2 \right\rangle$$

is the coefficient of the "super"-diffusion. Actually we have a matrix of the coefficients of the "super"-diffusion,

$$\hat{D} = \left(\left\langle \frac{1}{2} \lim_{t \to \infty} \frac{x_l(t) - x_l(0)}{(t \ln t)^{1/2}} \lim_{t \to \infty} \frac{x_m(t) - x_m(0)}{(t \ln t)^{1/2}} \right\rangle \right)_{l, m = 1, 2}$$

By (8.2), (8.3)

$$\hat{D} = \left(\frac{1}{2\tau} \sum_{j=1}^{N} \sum_{k=1}^{N_j} \alpha_{jk} \omega_{jkl} \omega_{jkm} \right)_{l, m = 1, 2}$$

and

$$D = \frac{1}{4\tau} \sum_{j=1}^{N} \sum_{k=1}^{N_j} \alpha_{jk}$$

Similarly, (8.1) shows that the diffusion coefficient in discrete dynamics is infinite as well. It is noteworthy that there is a sort of duality between the behavior of the diffusion coefficients in continuous-time and discrete dynamics: For the continuous-time dynamics the diffusion coefficient turns out to be infinite because of the slow decrease of the velocity autocorrelation function [see (1.8)], while for the discrete dynamics it is infinite because of the infinite variance of the free motion vector and not because of the slow decrease of the correlation function of the free motion vectors. This explains a discrepancy between the properties of the diffusion coefficients in the continuous-time and discrete dynamics which was noticed in ref. 16. Furthermore, one may introduce the coefficient of "super"-diffusion in the discrete dynamics as

$$D^0 = \frac{1}{4} \left\langle \left| \lim_{n \to \infty} \frac{\mathbf{x}_n - \mathbf{x}_0}{(n \ln n)^{1/2}} \right|^2 \right\rangle$$

By (8.1), (8.3)

$$D^0 = \frac{1}{4} \langle \xi_1^2 + \xi_2^2 \rangle = \frac{1}{4} \sum_{j=1}^{N} \sum_{k=1}^{N_j} \alpha_{jk} \tag{8.5}$$

Let us consider some particular cases.

Square Lattice. Consider a square lattice with unit space and let the scatterers be circles of radius $a < 1/2$ centered at the sites of the lattice. Because of the square lattice symmetry the covariance matrix

$$\langle \xi_l \xi_m \rangle = 2D^0 \delta_{lm}$$

is diagonal. If $a \geqslant \sqrt{2}/4$, there are only two pairs of basic corridors which are parallel to the x and y axes (we have one basic scatterer and two basic corridors in the direction of each axis which touch the basic scatterer from different sides). In that case

$$D^0 = \frac{1}{8\pi a} 2 \cdot 2(1-2a)^2 = \frac{1}{2\pi a}(1-2a)^2$$

If $a < \sqrt{2}/4$, there are more basic corridors and with a decrease of a more and more new basic corridors appear. Let us consider the limit of low density, $a \to 0$.

By Proposition 2.1 the direction $\boldsymbol{\omega}_{jk} = (\omega_{jk1}, \omega_{jk2})$ of any corridor is rational, so that $\omega_{jk2}/\omega_{jk1} = p/q$. We are interested in knowing for which p/q such a corridor does exist. Because of the symmetry of the square lattice, we may assume that $p \geqslant 0$; $q > 0$. We may also assume that p, q are relatively prime, $(p, q) = 1$ [we suppose that $(0, 1) = 1$ while $(0, q) \neq 1$, $q > 1$]. Let us project all the scatterers onto the y axis in parallel to the vector (q, p). Then a corridor in the direction (q, p) exists iff under this projection the images of all the scatterers do not cover the y axis entirely. The projection of an integer point (n, m) is $m - n(p/q) = r/q$, and the projection of a scatterer with a center at (n, m) is a segment of length $l = 2a(p^2 + q^2)^{1/2}/q$ with a center at r/q. So the corridor in the direction (q, p) exists iff $2a(p^2 + q^2)^{1/2}/q < 1/q$, or

$$(p^2 + q^2)^{1/2} < \frac{1}{2a}$$

The width of this corridor is equal to

$$d_{pq} = \frac{1}{(p^2 + q^2)^{1/2}} - 2a \tag{8.6}$$

and the (minimal) period of the lattice along the corridor is equal to

$$L_{pq}=(p^2+q^2)^{1/2} \tag{8.7}$$

According to (8.5),

$$D^0=\frac{1}{4}\sum_{k=1}^{N_c}\alpha_k \tag{8.8}$$

where N_c is the number of basic corridors and we write α_k instead of α_{jk} since we have now only one basic scatterer. Actually it is more convenient to index each corridor by integer numbers p, q where the vector (q, p) is parallel to the direction of the corridor. [As before, we have for each vector (q, p) a pair of basic corridors which touch the basic scatterer from different sides. This gives an additional factor 2 in subsequent formulas.] So we rewrite the last formula as

$$D^0=\sum_{(p,q)\in S_a}\alpha_{pq}$$

where the summation goes over the set

$$S_a=\left\{(p, q)\,|\,q>0;\, p\geqslant 0;\, (p, q)=1;\, (p^2+q^2)^{1/2}<\frac{1}{2a}\right\} \tag{8.9}$$

By (8.4), (8.6), and (8.7) we get that

$$\begin{aligned}D^0&=\sum_{(p,q)\in S_a}\alpha_{pq}=\frac{1}{4\pi a}\sum_{(p,q)\in S_a}2L_{pq}d_{pq}^2\\&=\frac{1}{2\pi a}\sum_{(p,q)\in S_a}(p^2+q^2)^{1/2}\left(\frac{1}{(p^2+q^2)^{1/2}}-2a\right)^2\end{aligned} \tag{8.10}$$

We have not used yet that a is small. The inclusion-exclusion formula gives that for $a\to 0$,

$$\sum_{(p,q)\in S_a}1=\frac{\pi}{4}\cdot\left(\frac{1}{2a}\right)^2\left[\prod_p\left(1-\frac{1}{p^2}\right)\right][1+O(a^{1/4})]$$

where the product goes over all prime numbers. By the Euler formula

$$\prod_p\left(1-\frac{1}{p^2}\right)=\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right)^{-1}=\frac{6}{\pi^2}$$

so

$$\sum_{(p,q)\in S_a}1=\frac{\pi}{4}\cdot\left(\frac{1}{2a}\right)^2\frac{6}{\pi^2}[1+O(a^{1/4})]=\frac{3}{8\pi a^2}[1+O(a^{1/4})] \tag{8.11}$$

Similar calculations give also that

$$\sum_{(p,q)\in S_a} (p^2+q^2)^{1/2} = \frac{1}{4\pi a^3}\,[1+O(a^{1/4})] \tag{8.12}$$

and

$$\sum_{(p,q)\in S_a} \frac{1}{(p^2+q^2)^{1/2}} = \frac{3}{4\pi a}\,[1+O(a^{1/4})] \tag{8.13}$$

From (8.11)–(8.13) we have that

$$\sum_{(p,q)\in S_a} (p^2+q^2)^{1/2}\left(\frac{1}{(p^2+q^2)^{1/2}}-2a\right)^2 = \frac{1}{4\pi a}\,[1+O(a^{1/4})]$$

so by (8.10),

$$D_0 = \frac{1}{8\pi^2 a^2}\,[1+O(a^{1/4})] \tag{8.14}$$

We will show in the Appendix that the mean length of free motion satisfies the estimates $C_1 a^{-1} < \tau < C_2 a^{-1}$, where $C_2 > C_1 > 0$ are absolute constants, so the coefficient of "super"-diffusion $D = (1/\tau)\, D^0$ satisfies the estimates

$$Ca^{-1} < D < C'a^{-1} \tag{8.15}$$

where $C' > C > 0$ are absolute constants.

Triangular Lattice. Let the scatterers be circles of radius $a < 1/2$ centered at the sites of the triangular lattice with unit space. Let $\mathbf{e}_1$, $\mathbf{e}_2$ be a basis of the lattice, $|\mathbf{e}_1| = |\mathbf{e}_2| = 1$, $(\mathbf{e}_1, \mathbf{e}_2) = 1/2$. If $a \geqslant \sqrt{3}/4$, there are no corridors and the system has a finite horizon. Let $a < \sqrt{3}/4$. Because of the triangular lattice symmetry the covariance matrix $\{\xi_l \xi_m\rangle = 2D^0 \delta_{lm}$ is again diagonal. To compute D^0, we use the same scheme as we did for the square lattice and we get the following expression:

$$D^0 = \frac{3}{4\pi a} \sum_{(p,q)\in T_a} (p^2+pq+q^2)^{1/2}\left(\frac{\sqrt{3}}{2(p^2+pq+q^2)^{1/2}} - 2a\right)^2$$

where

$$T_a = \left\{(p,q)\,|\, q>0;\, p\geqslant 0;\, (p,q)=1;\, |q\mathbf{e}_1 + p\mathbf{e}_2| \right.$$
$$\left. = (p^2+pq+q^2)^{1/2} < \frac{\sqrt{3}}{4a}\right\}$$

For $a \to 0$,

$$D^0 = \frac{25\sqrt{3} - 36}{64\pi^2 a^2} [1 + O(a^{1/4})]$$

Again, as for the square lattice, the mean length of free motion satisfies the estimates $C_1 a^{-1} < \tau < C_2 a^{-1}$, so for the coefficient of "super"-diffusion we have the estimates (8.15).

APPENDIX. AN ESTIMATE OF THE MEAN LENGTH OF FREE MOTION

Let us consider a square lattice of circular scatterers of radius $a > 0$. Our aim is to prove that for $a \to 0$ the mean length of free motion $\tau = \langle |\mathbf{r}(\lambda)| \rangle$ satisfies the estimates

$$C_1 a^{-1} < \tau < C_2 a^{-1} \tag{A1}$$

where $C_2 > C_1 > 0$ are absolute constants. Without proof this was stated in ref. 9. Our proof is based on the following lemma.

Lemma A1. Any segment $[\mathbf{x}, \mathbf{y}]$ of free motion of length $|\mathbf{x} - \mathbf{y}| > 100a^{-1}$ belongs to a corridor.

Proof. Let $\mathbf{y} - \mathbf{x} = \mathbf{r} = (r_1, r_2)$. We may assume that $0 \leqslant r_2 \leqslant r_1$ and $\omega = r_2/r_1$ is irrational. Let us expand ω in a continued fraction, $\omega = [a_1, a_2, \ldots]$. Denote

$$\frac{p_n}{q_n} = [a_1, \ldots, a_2]$$

Let us choose n in such a way that

$$q_n \leqslant 3a^{-1} < q_{n+1} \tag{A2}$$

For the sake of definiteness we will assume that $p_n/q_n > \omega$. Then

$$\frac{p_n}{q_n} > \omega > \frac{p_{n+1}}{q_{n+1}} = \frac{p_{n-1} + a_{n+1} p_n}{q_{n-1} + a_{n+1} q_n} > \frac{p_{n-1} + (a_{n+1} - 1) p_n}{q_{n-1} + (a_{n+1} - 1) q_n}$$
$$> \cdots > \frac{p_{n-1}}{q_{n-1}} \tag{A3}$$

and

$$0 < \frac{p_n}{q_n} - \omega < \frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}} = \frac{1}{q_n q_{n+1}} < \frac{a}{3q_n} \tag{A4}$$

Let $\mathbf{x} \in \partial\Omega_\alpha$, $\mathbf{y} \in \partial\Omega_\beta$, and

$$\Omega_\gamma = T_1^{q_n} T_2^{p_n} \Omega_\alpha, \qquad \Omega_\delta = T_1^{q_{n-1}} T_2^{p_{n-1}} \Omega_\alpha, \qquad \Omega_\varepsilon = T_1^{q_n} T_2^{p_n} \Omega_\delta$$
$$= T_1^{q_n + q_{n-1}} T_2^{p_n + p_{n-1}} \Omega_\alpha$$

Consider a line l_1 which is tangent to Ω_α, Ω_γ from below and a line l_2 which is tangent to Ω_δ, Ω_ε from above (see Fig. 4). Then l_1 and l_2 are parallel and their incline is equal to p/q.

We state that: (i) l_1 lies above l_2; (ii) the strip between l_1 and l_2 is a corridor; (iii) $[\mathbf{x}, \mathbf{y}]$ belongs to that corridor. To prove (i), let us remark that the inequalities (A3) imply that the segment $[\mathbf{x}, \mathbf{y}]$ passes between Ω_δ and Ω_γ, so l_1 does lie above l_2. Let us prove (ii).

Let C be the strip between l_1 and l_2. Assume that a scatterer Ω_λ exists which intersects C. Because of the periodicity of the lattice of scatterers in the direction of C, we may assume that Ω_λ lies between Ω_δ and Ω_ε in the sense that the projection of the center of Ω_λ onto the line l_1 lies between the projections of the centers of Ω_δ and Ω_ε onto this line. Let us denote the center of a scatterer Ω_ξ by O_ξ, $\xi = \alpha, \beta, \ldots$. Let $\Omega_\lambda = T_1^q T_2^p \Omega_\alpha$. It is clear from Fig. 4 that O_λ lies inside the angle $\angle O_\delta O_\alpha O_\gamma$, so

$$\frac{p_{n-1}}{q_{n-1}} - \omega \leqslant \frac{p}{q} - \omega \leqslant \frac{p_n}{q_n} - \omega$$

and $q < q_n + q_{n-1}$. This means that either p/q gives a better approximation of ω from above than p_n/q_n does or p/q gives a better approximation of ω from below than p_{n-1}/q_{n-1} does. Since both are impossible for $q < q_n + q_{n-1}$, we have proved (ii).

To prove (iii), let us assume that $[\mathbf{x}, \mathbf{y}]$ passes between scatterers $(T_1^{q_n} T_2^{p_n})^j \Omega_\delta$ and $(T_1^{q_n} T_2^{p_n})^{j+1} \Omega_\delta$, $j \geqslant 1$. Then, a simple geometrical calculation shows

$$\tan \eta > \frac{2a}{(p_n^2 + q_n^2)^{1/2}} \tag{A5}$$

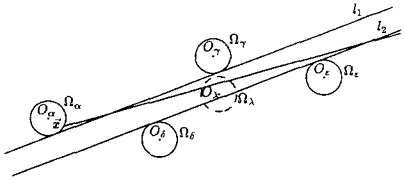

Fig. 4. An illustration for the proof of Lemma A1.

where η is the angle between the segment $[\mathbf{x}, \mathbf{y}]$ and the line l_1. But from (A4) it follows easily that

$$\eta < \frac{a}{3q_n} \leqslant \frac{a\sqrt{2}}{3(p_n^2+q_n^2)^{1/2}} \tag{A6}$$

A contradiction between (A5) and (A6) proves that $[\mathbf{x}, \mathbf{y}]$ cannot pass between scatterers $(T_1^{q_n}T_2^{p_n})^j\,\Omega_\delta$ and $(T_1^{q_n}T_2^{p_n})^{j+1}\,\Omega_6$, $j \geqslant 1$. Using the same arguments as in the proof of point (ii) above, we get that actually $y \in (T_1^{q_n}T_2^{p_n})^j\,\Omega_\delta$ for some $j \geqslant 1$. Lemma A1 is proved.

Let us prove now the estimates (A1). Let

$$W = \{\lambda \in \Lambda_0 \mid |\mathbf{r}(\lambda)| > Ma^{-1}\}$$

where $M > 100$ is a large constant which will be chosen later. We have

$$\tau = \langle |\mathbf{r}(\lambda)| \rangle = \int_{\Lambda_0 \setminus W} |\mathbf{r}(\lambda)|\, \mu_0(d\lambda) + \int_W |\mathbf{r}(\lambda)|\, \mu_0(d\lambda)$$

$$\leqslant Ma^{-1} + \int_W |\mathbf{r}(\lambda)|\, \mu_0(d\lambda) \tag{A7}$$

To estimate $\int_W |\mathbf{r}(\lambda)|\, \mu_0(d\lambda)$, let us remark that if $\lambda \in W$, then

$$|\mathbf{r}(\lambda)| \geqslant Ma^{-1} > 100a^{-1}$$

so by Lemma A1, $[\mathbf{x}(\lambda), \mathbf{x}(T\lambda)]$ belongs to some corridor C_k, so that $W = \bigcup_{k=1}^{N_c} W_k$, where

$$W_k = \{\lambda \in W \mid [\mathbf{x}(\lambda), \mathbf{y}(\lambda)] \in C_k\}$$

By Proposition 4.2 we have the basic formula

$$\Pr\{\lambda \in W_k,\ |\mathbf{r}(\lambda)| \geqslant R,\ \pm(\mathbf{r}(\lambda), \omega_k) > 0\} = \frac{\alpha_k^{\pm}}{R^2}\,[1 + \varepsilon_k^{\pm}(R)] \tag{A8}$$

where $\varepsilon_k^{\pm}(R) = O(R^{-1/2})$ as $R \to \infty$. An inspection of the proof of Proposition 4.2 shows that an absolute constant $M > 0$ exists such that for $R > Ma^{-1}$,

$$\frac{C_0}{(aR)^{1/2}} \leqslant |\varepsilon_k^{\pm}(R)| \leqslant \frac{C_1}{(aR)^{1/2}}$$

where $C_1 > C_0 > 0$ are absolute constants. Hence

$$C_2 \alpha_k a \leqslant \int_{W_k} |\mathbf{r}(\lambda)|\, \mu_0(d\lambda) \leqslant C_3 \alpha_k a$$

where $C_3 > C_2 > 0$ are absolute constants, and

$$C_2\left(\sum_{k=1}^{N_c} \alpha_k\right) a \leqslant \int_W |\mathbf{r}(\lambda)|\ \mu_0(d\lambda) \leqslant C_3\left(\sum_{k=1}^{N_c} \alpha_k\right) a$$

By (8.8), (8.14)

$$\sum_{k=1}^{N_c} \alpha_k = \frac{1}{2\pi^2 a^2}\,[1 + O(a^{1/4})]$$

so

$$C^{(0)} a^{-1} \leqslant \int_W |\mathbf{r}(\lambda)|\ \mu_0(d\lambda) \leqslant C^{(1)} a^{-1}$$

where $C^{(1)} > C^{(0)} > 0$ are absolute constants. Hence we have from (A7) that

$$C^{(0)} a^{-1} \leqslant \tau \leqslant (C^{(1)} + M)\, a^{-1}$$

Estimate (A1) is proved.

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